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CRITICAL CONFIGURATIONS (DETERMINANTAL LOCI) FOR RANGE AND RANGE-DIFFERENCE SATELLITE NETWORKS

By

Emmanuel Tsimis

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PREFACE

This project in under the supervision of Ivan I. Mueller, Professor of the Department of Geodetic Science at The Ohio State University, and under the technical direction of James P. Murphy, Code ES, NASA Headquarters, Washington, D. C. The contract is administered by the Office of University Affairs, NASA, Washington, D. C. 20546.

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INTRODUCTION

The present work comprises basically three essays, one for each observational mode of Geometric Satellite Geodesy.

The first one, published separately as No. 185 in this Report series, "On the Geometric Analysis and Adjustment for Optical Satellite Observations," is about the optical observations mode. In that, the geometrical analysis of the problem yielded a well-known regression model for the adjustment of the observations along with a suitable and convenient metric for the least-squares optimum criterion. In the present treatment of the problem, considerable emphasis has been placed on the geometry, and no additional element has been introduced for the adjustment.

The second essay, which is in this Report (Section 1), is about the determinantal loci (critical configurations) for range networks. This problem has been treated before in (Rinner [1966]) and (Blaha [1971]), the latter being an extension of the former along similar guidelines.

The present treatment approaches the topic from the viewpoint of analysis. An attempt is made to found the results theoretically by using elements of the theory of invariants. To the writer's opinion the present treatment is not the most elegant one. Some partial results from an attempt for a pure geometrical approach, which remained unfinished, hint that a geometrical approach would be more elegant.

The third essay, also in this Report (Section 2), is about the determinantal loci for networks of stations observing range differences (e.g., by means of geoceivers). In order to find out these determinantal loci, an analytical way is followed, similar to that used for the case of range networks. Although the problem for "continuously" measured range differences presents some similarities in treatment with that of ranges, it is different from that in many aspects.

A number of appendices is included which constitute a substantial part of the whole work.

As for the mathematical language used, it can be said that formal mathematical language has been avoided.

1. DETERMINANTAL LOCI FOR RANGE NETWORKS

1.1 Introductory Remarks

1.1.1 Recovery of the Satellite-Position Points

In the range observations mode each participating station P_i at an event $[E_j, Q_j | t_j]$ observes the length of the distance (P_iQ_j), i.e.

$$r_{ij} = (\overline{P_iQ_j}).$$

This distance determines a spherical surface with center at P_i and radius r_{ij}, which may be considered as a locus of the satellite-position point Q_j. Thus Q_j is determined as the intersection of spherical surfaces with centers at the station-position points P_i and radii the lengths of the distances of Q_j from P_i. In order to formulate it analytically, let $[x_i, y_i, z_i]$ be the Cartesian coordinates of P_i and $[X_j, Y_j, Z_j]$ of Q_j, with respect to a coordinate system tied to the solid earth. Then for the determination of Q_j, one has the system of equations of the form:

$$(X_j-x_i)^2+(Y_j-y_i)^2+(Z_j-z_i)^2-r_{ij}^2 = 0, \quad (i=1,2,\ldots) \tag{1}$$

where $[x_i, y_i, z_i]$ are considered to be known here, and the range of the index i remains to be determined for the unique determination of $Q_j(X_j, Y_j, Z_j)$. From the implicit function theorem [see Appendix A] it is inferred that a system of three equations of the form (1) may be solved for X_j, Y_j, Z_j in terms of x_i, y_i, z_i, (i=1, 2, 3) provided certain conditions are satisfied. Thus if the system,

$$
\begin{aligned}
(X_j-x_1)^2+(Y_j-y_1)^2+(Z_j-z_1)^2-r_{1j}^2 &= 0\\
(X_j-x_2)^2+(Y_j-y_2)^2+(Z_j-z_2)^2-r_{2j}^2 &= 0\\
(X_j-x_3)^2+(Y_j-y_3)^2+(Z_j-z_3)\ -r_{3j}^2 &= 0
\end{aligned}
\tag{2}
$$

does have at least one solution-point* and

$$
\begin{vmatrix}
X_j-x_1 & Y_j-y_1 & Z_j-z_1\\
X_j-x_2 & Y_j-y_2 & Z_j-z_2\\
X_j-x_3 & Y_j-y_3 & Z_j-z_3
\end{vmatrix}
\neq 0,
\tag{3}
$$

then it can be solved for X_j, Y_j, Z_j in terms of x_i, y_i, z_i, r_{ij}, $(i=1,\ 2,\ 3)$, in the neighborhood of a solution-point, (a point which annuls (2).

Although the implicit function theorem assures the feasibility of the solution of the system (2) with respect to X_j, Y_j, Z_j, it does not explain the set of the solution-points. This is in fact an algebraic geometry problem, and one may take the occasion to say that the whole deal about range and range-difference networks in the present work belongs to the jurisdiction of algebraic geometry.

The set of solution-points of (2) is treated in Appendix B, where it is demonstrated that under condition (B15) a set of points which annul (2) exists, and under (3), that set contains two points symmetric with respect to the plane of the centers. Thus in order to tie the satellite-position points to the solid earth, three known stations observing simultaneously are necessary and sufficient to do that. Therefore, in a series

---

* It is noted that here and throughout this work the field of concern is that of real numbers.

of range observations events where three stations observe simultaneously, one has no redundant* information at all. For redundant information at least four stations observing simultaneously are necessary, provided their configuration is not a degenerate one. When more than four stations observe simultaneously, they are sufficient but not necessary for redundant information from the observations.

1.1.2 Introduction of the Coordinate System

Suppose that there are m stations, P_i, ($i = 1,2, \ldots, n$) observing simultaneously a series of range observations events $[E_j, Q_j \mid t_j]$, ($j = 1,2, \ldots, m$). The configuration of the points P_i, Q_j is a polyhedron, not necessarily convex, in the extended field of geometric geodesy. The polyhedron must be "built" through the observations. One may consider that polyhedron being constructed by piecing together tetrahedra, as building blocks. To start constructing the polyhedron, a basis must be taken upon which the which the rest is founded. It is desirable for the basis to be characterized as necessary and sufficient for this purpose. Such a basis is the 3-dimensional simplex, that is, the tetrahedron.

* To put it formally, redundant information from a set of satellite observations for geometric satellite geodesy, is that which is beyond the necessary and sufficient for the unique determination of the satellite position points, (extension points of the field of geometric geodesy).

The coordinate system to be introduced, is attached to the tetrahedron that serves as basis. This is usually done by considering a coordinate system and fixing certain coordinates in such a way that a non-degenerate tetrahedron is determined. If for an example, one constrains the three coordinates of P_1, two of the coordinates of P_2 and one of the coordinates of P_3, i.e.,

$$x_1,\ y_1,\ z_1 = \text{constant}$$

$$y_2,\ z_2 = \text{constant}$$

$$z_3 = \text{constant}$$

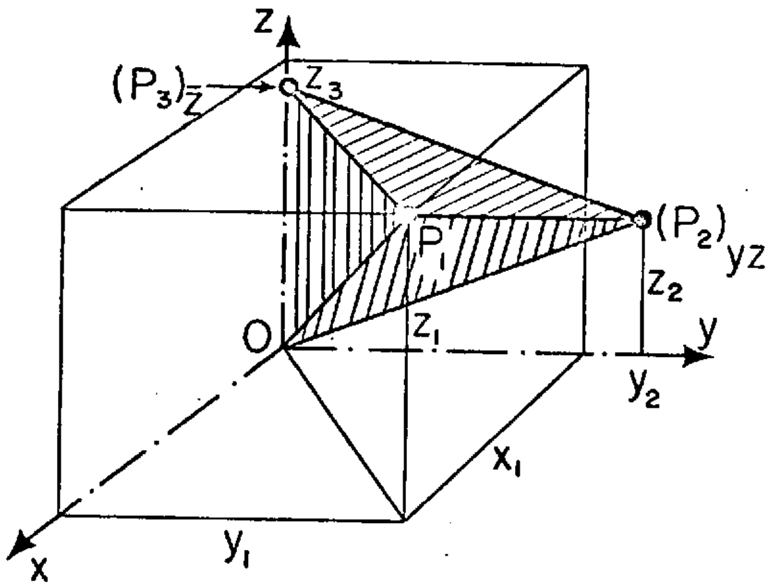

Figure 1

the fixed tetrahedron is formed from the points (see Figure 1):

0 (o,o,o)	- origin of the coordinate system,
P_1 (x_1, y_1, z_1)	- the point, the coordinates of which are all fixed,
$(P_2)_{yz}$ $(0, y_2, z_2)$	- the projection on the yz-coordinate plane of the point P_2 whose coordinates y and z are fixed,
$(P_3)_z$ $(0, 0, z_3)$	- the projection on the z-axis of the point P_3 whose coordinate z is fixed.

Some people choose $x_1 = y_1 = z_1 = y_2 = z_2 = z_3 = 0$. For numerical calculations this choice is preferable; however, in a theoretical treatment it is not advisable, for it destroys the symmetry of the expressions, and consequently deprives one from a great advantage for elegant manipulations. Therefore in the following treatment, the constrained coordinates will be left unspecified as long as there is no need to do so.

1.1.3 Minimum Number of Events for a Unique Solution.

Consider now the configuration of the points

P_i, $(i = 1, 2, \ldots, n)$, $\quad Q_j$, $(j = 1, 2, \ldots, m)$, and let

$$P_i \equiv (x_i \;\; y_i, z_i), \qquad i = 1, 2, \ldots, n$$

$$Q_j \equiv (X_j, Y_j, Z_j), \qquad j = 1, 2, \ldots, m,$$

be the analytic representations of the points P_i, Q_j with respect to a cartesian coordinate system, which has been introduced through an admissible set of constraints, say those quoted above, i.e.,

$x_1, y_1, z_1, y_2, z_2, z_3$ = constants.

Suppose that except the constrained coordinates, all the others are unknown. Thus there are $(3n-6)$ unknown station-coordinates. One recalls that $n\geq4$ and then $(3n-6)\geq6$. A question of interest in the following discussion is the minimum number of events for a unique determination of the unknown station-coordinates. At each event three unknown quantities are introduced, namely the coordinates of the satellite-position point, while n equations result from the simultaneous observations of the n stations, which are of the form (1). It was proved above that three equations are, (under certain conditions), necessary and sufficient for the determination of the satellite-position point. Therefore at each event $(n-3)$ equations are redundant. For a unique determination of the $3n-6$ unknown station-coordinates, one has to find out min m under the condition

$$m(n-3) \geq 3n-6, \quad n \geq 4. \tag{4}$$

That is

$$\min m = \begin{cases} 3\frac{(n-2)}{(n-3)}; & \text{if } 3\frac{(n-2)}{(n-3)} \text{ is an integer} \\ \\ [3\frac{(n-2)}{(n-3)}]+1; & \text{if } 3\frac{(n-2)}{(n-3)} \text{ is not an integer,} \end{cases} \tag{5}$$

where $[\frac{3(n-2)}{(n-3)}]$ stands for the integral part of $3(n-2)/(n-3)$.

Although n may be at least theoretically, any positive integer ≥ 4, this is not the case for m, for the sequence $q_n = 3(n-2)/(n-3)$ decreases monotonicly, converging to 3 as $n \to oo$, i.e.,

$$\lim_{n \to oo} \frac{3(n-2)}{(n-3)} = 3$$

Thus one has

$$\begin{aligned} \min m &= 6 \quad \text{for} \quad n = 4 \\ &= 5 \quad \text{"} \quad n = 5 \\ &= 4 \quad \text{"} \quad n > 5, \text{(i.e. } n = 6, 7, 8, \ldots). \end{aligned} \tag{6}$$

1.1.4 <u>Patterns of Observations</u>

By pattern of observation, it is meant the way a set of stations observes a set of satellite-position points. For a given number of stations, each observational pattern is a combination of some <u>basic</u> observational patterns. In the following, a formula will be set up, giving the number of the basic observational patterns of any number of stations. Although such a formula does not seem to possess any practical interest, trying to obtain it one gains an insight to the situation.

Let there exist n stations. It was shown above that the minimum number of stations participating in an event must be four. Therefore the n stations may observe either in groups of 4, or 5, ..., or (n-1) or all together. The total number of basic observational patterns then is

$$N = C_4^n + C_5^n + \cdots + C_{n-1}^n + C_n^n$$

$$= 2^n - (C_0^n + C_1^n + C_2^n + C_3^n)$$

$$= 2^n - (1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6})$$

or

$$N = 2^n - (n+1)\cdot(1+\frac{n(n-1)}{6}) \tag{7}$$

The last formula gives

$$\begin{aligned} N &= 1 && \text{for} \quad n = 4 \\ &= 6 && \text{"} \quad n = 5 \\ &= 22 && \text{"} \quad n = 6 \\ &= 64 && \text{"} \quad n = 7 \\ &= 163 && \text{"} \quad n = 8 \\ &\ldots && \ldots \quad \ldots \end{aligned} \tag{8}$$

1.2 Definition of Determinantal Loci

Consider a set of n stations $P_i(z = 1, 2, \ldots, n)$ observing in some pattern m events, corresponding to the satellite-position points Q_j, $(j = 1, 2, \ldots, m)$. Think of the satellite-position point Q_j being observed simultaneously by n_j stations, where $n_j \geq 4$. Then for each Q_j there exist n_j observation equations of the form (1), i.e.

$$\begin{aligned} fji_1 &\equiv (X_j - x_{i_1})^2 + (Y_j - y_{i_1})^2 + (Z_j - z_{i_1})^2 - r_{i_1 j}^2 = 0, \\ fji_2 &\equiv (X_j - x_{i_2})^2 + (Y_j - y_{i_2})^2 + (Z_j - z_{i_2})^2 - r_{i_2 j}^2 = 0, \\ &\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ fji_{n_j} &\equiv (X_j - x_{i_{n_j}})^2 + (Y_j - y_{i_{n_j}})^2 + (Z_j - z_{i_{n_j}})^2 - r_{i_{n_j} j}^2 = 0, \end{aligned} \tag{9}$$

where $i_1, i_2, \ldots, i_{n_j}$ are indices from the set $\{1, 2, 3, \ldots, n\}$ which correspond to the stations observing simultaneously Q_j, and $j = 1, 2, 3, \ldots, m$. Totally there are

$$\nu = n_1+n_2+n_3+\ldots+n_j+\ldots+n_m \tag{10}$$

observation equations, and

$$\mu = (3n+3m)-6 = 3(n+m-2) \tag{11}$$

unknowns, namely X_j, Y_j, Z_j, $(j=1,\ldots,m)$ and x_i, y_i, z_i, $(i=1,..,n)$ with six of them constrained. It is supposed that $\nu\geq\mu$; otherwise the system is under-determined. Consider the system of μ equations out of ν. From the implicit function theorem one has that if a solution point to this system exists*, then it can be solved uniquely for the μ unknown variables in a neighborhood of that solution-point, provided the jacobian of the functions of the left-hand parts of this system does not vanish in that neighborhood, (see Appendix A, relation A3). Since one may choose μ equations out of ν in as many ways as $C_\mu^\nu = \nu!/\mu!(\nu-\mu)!$, in order to include all these cases, the above condition for the jacobian is stated as follows. The jacobian matrix of the functions of the left-hand parts of the system (9) (ν equations in μ unknowns) is of rank μ, i.e.,

$$\text{rank}\left(\frac{\partial(fji_1,\ldots fji_{n_j})}{\partial(X_j,Y_j,Z_j,x_i,y_i,z_i)}\right) = \underset{\substack{i=1,2,\ldots,n\\ j=1,2,\ldots,m}}{\text{rank } J_M=\mu=3(n+m-2)} \tag{12}$$

If the rank of the above jacobian matrix is less than μ, one cannot find μ independent functions out of ν, that means that additional relations exist between the variables. Since the rank of the jacobian matrix is an invariant under linear transformations (see Appendix C), the additional relations between the variables have to be invariants too, under

*One recalls that in the implicit function theorem (see Appendix A) the existence of a solution-point is assumed.

linear transformations, and therefore they represent geometric relations (see [Tsimis, 1972] section 2.2.2.3). The totality of those additional relations, (for which the jacobian matrix is of rank less than μ) constitute the determinantal loci.* Therefore in order to find out the determinantal loci, one has to examine when the jacobian matrix is of rank less than μ. Although the jacobian matrix is huge, $(\nu x \mu)$, each of its rows has at most six non-zero elements out of μ, and when one takes the determinants of the submatrices $\mu x \mu$ to check for the rank, pivotal condensation techniques reduce those determinants considerably (see Appendices D and E).

1.3 Determinantal Loci and Adjustment of the Observations

Consider again the system (9) being represented by one of its equations, in order to simplify the notation, i.e.,

$$f_{ji} = (X_j-x_i)^2+(Y_j-y_i)^2+(Z_j-z_i)^2-r_{ij}^2 = 0 \tag{13}$$

where

$$\begin{aligned} &j\varepsilon\{1,2,\ldots,m\} \\ &i\varepsilon\{i_1,i_2,\ldots,i_{n_j}\} \quad \{1,2,3,\ldots,n\}. \end{aligned} \tag{14}$$

By considering r_{ij} as unknowns in addition to X_j, Y_j, Z_j, x_i, y_i, z_i, take the differential form at (13). Then

$$2(X_j-x_i)(dX_j-dx_i)+2(Y_j-y_i)(dY_j-dy_i)+2(Z_j-z_i)(dZ_j-dz_i) = 2r_{ij}dr_{ij} \tag{15}$$

or

$$\frac{(X_j-x_i)}{r_{ij}}(dX_j-dx_i) + \frac{(Y_j-y_i)}{r_{ij}}(dY_j-dy_i) + \frac{(Z_j-z_i)}{r_{ij}}(dZ_j-dz_i)=dr_{ij}, \tag{16}$$

---

*The name is justified from the fact that each of these additional relations annuls the determinants of all $\mu x \mu$ submatrices of the jacobian matrix.

Refer this differential form to the point $(X_j^o, Y_j^o, Z_j^o, x_i^o, y_i^o, z_i^o, r_{ij}^{oo})$ and let

$$dr_{ij} = r_{ij}-r_{ij}^o = (r_{ij}-r_{ij}^b)-(r_{ij}^o-r_{ij}^b) = v_{ij}-l_{ij}$$

where r_{ij} is the true value, r_{ij}^b is the observed one $v_{ij} = r_{ij}-r_{ij}^b$ and $l_{ij} = r_{ij}^o-r_{ij}^b$.

Then (16) yields

$$\frac{(X_j^o-x_i^o)}{r_{ij}^o}(dX_j-dx_k)+\frac{(Y_j^o-y_i^o)}{r_{ij}^o}(dY_j-dy_i)+\frac{(Z_j^o-z_i^o)}{r_{ij}^o}(dZ_j-dz_i) = v_{ij}-l_{ij}, \quad (17)$$

or in matrix form

$$L = -AX+V \quad (18)$$

where

$${}_{\nu}L_1 = (l_{ij}), \quad {}_{\nu}V_1 = (v_{ij}), \quad {}_{(\mu+6)}X_1 = (dX_j, dY_j, dZ_j, dx_i, dy_i, dz_i)^T$$

and ${}_{\nu}A_{(\mu+6)}$ results from the jacobian matrix J_M (see relation [12]), by dividing each of its rows by the corresponding r_{ij}^o. It is noted that since $r_{ij}\neq 0$ the rank of J_M does not change by that operation and consequently

$$\text{rank } A = \text{rank } J_M. \quad (19)$$

Under the assumptions $E(V) = 0$ and $Cov(V) = E(VV^T) = \sigma^2 I$ where σ^2 is unknown, relation (18) is a linear statistical model. From the theory of the linear model one has that the best (minimum-variance) linear (linear functions of l_{ij}) unbiased estimate of X is given by least squares and is unique iff rank $A = \mu$. By recalling relations (19) and (12) one concludes that a unique least squares solution exists iff

$$\text{rank } A = \text{rank } J_M = \mu,$$

i.e., if there exist no determinantal loci. That is why the determinantal loci are studied in the present work.

1.4 Determinantal Loci of Specified Observational Patterns

1.4.1 General Remarks

In this section specified observational patterns will be examined in order to find out their determinantal loci. That is, given the number of stations and the way they observe (observational pattern), the corresponding jacobian matrix will be searched for the cases when its rank is less than the number of the unknowns. Despite the fact that there exist a large number of cases to be examined, a systematic procedure may cover them all, at least by enumeration, for they constitute a denumerable set.

It is recalled [see section 1.1.1] that in general, four ground stations are necessary and sufficient for each event. This must especially be stressed, for it is the core of the whole problem. Thus the case of four ground stations is treated first. No specification of the coordinates which are to be constrained is made until a general formula is reached that is susceptible to all the specifications. The theory of determinants is used extensively; however, explicit expressions of the determinants are given instead of symbolic, sacrificing compactness for an easier understanding of the manipulations involved.

1.4.2 The Case of Four Ground Stations

Let $P_1 \equiv (x_1, y_1, z_1)$, $P_2 \equiv (x_2, y_2, z_2)$, $P_3 \equiv (x_3, y_3, z_3)$, $P_4 \equiv (x_4, y_4, x_4)$ be four simultaneously observing ground stations, and $Q_j \equiv (X_j, Y_j, Z_j)$, $(j = 1, 2, \ldots, m)$ be a set of satellite-position points observed by those stations. Then the functional relationships on which the statistical model is based, have the form

$$f_{ij} \equiv (X_j - x_i)^2 + (Y_j - y_i)^2 + (Z_j - z_i)^2 = r_{ij}^2$$

$$i = 1, 2, 3, 4 \tag{20}$$

$$j = 1, 2, \ldots, m, \quad m \geq 6$$

[see section 1.2 and 1.3).

This system contains

$$\nu = 4m \geq 24 \text{ equations} \tag{21}$$

and

$3m+3x4 = 3(m+4)$ unknown variables, six of which are to be constrained, so that there actually are

$$\mu = 3\,(m+2) \text{ unknown variables.} \tag{22}$$

Nevertheless, while taking the jacobian matrix (J_M) for the system (20), no unknown variable is considered constrained. Thus the jacobian matrix of the system (20) is of dimension

$$4m x 3(m+4).$$

According to what was said in section 1.3, the least squares solution is unique iff

$$\text{rank } J_M = \mu = 3(m+2) \tag{23}$$

while for

$$\text{rank } J_M < \mu = 3(m+2) \tag{24}$$

one has the determinantal loci of the configuration. Therefore in order to find out the determinantal loci, one has to consider the determinants of all $\mu x \mu$ submatrices of the jacobian matrix J_M, where the value of $\mu = 3(m+2)$ is determined from the value of m which is obtained by equating the number of unknowns to the number of equations, i.e.

$$4m = 3(m+2) \Rightarrow \quad m=6. \tag{25}$$

Thus $$\mu = 3(m+2) = 24 \tag{26}$$

It should be remembered that m=6 it the minimum number of events for a unique solution in the case of four ground stations, (see section 1.1.3, relation [6]). Manipulating with determinants of that order (i.e. 24-th) seems to be an appalling work; however they may be reduced in order by pivotal condensation in a systematic way, which leads to a sort of condensation at the jacobian matrix, being called here <u>condensed jacobian matrix</u>. In order to avoid interruption of the main objective by annoying details, the derivation of the condensed jacobian matrix is given in Appendix E, from where the final form is recalled in Table 1.

x_1	y_1	z_1	x_2	y_2	z_2
+	+	+	-	-	-
$(X_j-x_1)\Delta_1^j$	$(Y_j-y_1)\Delta_1^j$	$(Z_j-z_1)\Delta_1^j$	$(X_j-x_2)\Delta_2^j$	$(Y_j-y_2)\Delta_2^j$	$(Z_j-z_2)\Delta_2^j$

x_3	y_3	z_3	x_4	y_4	z_4
+	+	+	-	-	-
$(X_j-x_3)\Delta_3^j$	$(Y_j-y_3)\Delta_3^j$	$(Z_j-z_3)\Delta_3^j$	$(X_j-x_4)\Delta_4^j$	$(Y_j-y_4)\Delta_4^j$	$(Z_j-z_4)\Delta_4^j$

<u>Table 1.</u> The Condensed Jacobian Matrix for Four Stations

In Table 1:

$$\Delta_1^j = \begin{vmatrix} X_j-x_2 & Y_j-y_2 & Z_j-Z_2 \\ X_j-x_3 & Y_j-y_3 & Z_j-z_3 \\ X_j-x_4 & Y_j-y_4 & Z_j-z_4 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

$$\Delta_2^j = \begin{vmatrix} X_j-x_1 & Y_j-y_1 & Z_j-z_1 \\ X_j-x_3 & Y_j-y_3 & Z_j-z_3 \\ X_j-x_4 & Y_j-y_4 & Z_j-z_4 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \qquad (j = 1,2,\ldots,m), \quad (27)$$

$$\Delta_3^j = \begin{vmatrix} X_j-x_1 & Y_j-y_1 & Z_j-z_1 \\ X_j-x_2 & Y_j-y_2 & Z_j-z_2 \\ X_j-x_4 & Y_j-y_4 & Z_j-z_4 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \qquad (j=1,2,\ldots,m), \qquad (27)\ \text{(cont.)}$$

$$\Delta_4^j = \begin{vmatrix} X_j-x_1 & Y_j-y_1 & Z_j-z_1 \\ X_j-x_2 & Y_j-y_2 & Z_j-z_2 \\ X_j-x_3 & Y_j-y_3 & Z_j-z_3 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

(compare Appendix E, relations [E14]).

The above expression of the condensed jacobian matrix is a compact symmetric form, where the constrained coordinates are not specified. Each time a set of admissible constraints is selected, one has to delete the corresponding columns from the above expression. Because of the symmetry it is very flexible to specifications. The determinants of the 24x24 submatrices of the original jacobian matrix, which are taken for the determinantal loci, are equal to the determinants of the corresponding 6x6 submatrices of the condensed jacobian matrix. Therefore for the determinantal loci one has to consider the determinants of all 6x6 submatrices of the condensed jacobian matrix. To fix ideas, suppose that the coordinates $x_1, y_1, z_1, y_2, z_2, z_3$ are constrained. Deleting the corresponding columns from the condensed jacobian matrix, it becomes an mx6 array, i.e.,

x_2	x_3	y_3	x_4	y_4	z_4
$-(X_j-x_2)\Delta_2^j$	$+(X_j-x_3)\Delta_3^j$	$+(Y_j-y_3)\Delta_3^j$	$-(X_j-x_4)\Delta_4^j$	$-(Y_j-y_4)\Delta_4^j$	$-(Z_j-z_4)\Delta_4^j$

Table 2. The Condensed Jacobian Matrix for Four Stations When the coordinates x_1, y_1, z_1, y_2, z_2, z_3 are constrained

For the determinantal loci one must examine the zeroes of the determinants of the 6x6 submatrices. For a representative of these determinants, consider that of the first six rows, i.e.,

$$|\Delta|_{1,..,6}=\begin{vmatrix} -(X_1-x_2)\Delta_2^1 & (X_1-x_3)\Delta_3^1 & (Y_1-y_3)\Delta_3^1 & -(X_1-x_4)\Delta_4^1 & -(X_1-y_4)\Delta_4^1 & -(Z_1-z_4)\Delta_4^1 \\ -(X_2-x_2)\Delta_2^2 & (X_2-x_3)\Delta_3^2 & (Y_2-y_3)\Delta_3^2 & -(X_2-x_4)\Delta_4^2 & -(Y_2-y_4)\Delta_4^2 & -(Z_2-z_4)\Delta_4^2 \\ -(X_3-x_2)\Delta_2^3 & (X_3-x_3)\Delta_3^3 & (Y_3-y_3)\Delta_3^3 & -(X_3-x_4)\Delta_4^3 & -(Y_3-y_4)\Delta_4^3 & -(Z_3-z_4)\Delta_4^3 \\ -(X_4-x_2)\Delta_2^4 & (X_4-x_3)\Delta_3^4 & (Y_4-y_3)\Delta_3^4 & -(X_4-x_4)\Delta_4^4 & -(Y_4-y_4)\Delta_4^4 & -(Z_4-z_4)\Delta_4^4 \\ -(X_5-x_2)\Delta_2^5 & (X_5-x_3)\Delta_3^5 & (Y_5-y_3)\Delta_3^5 & -(X_5-x_4)\Delta_4^5 & -(Y_5-y_4)\Delta_4^5 & -(Z_5-z_4)\Delta_4^5 \\ -(X_6-x_2)\Delta_2^6 & (X_6-x_3)\Delta_3^6 & (Y_6-y_3)\Delta_3^6 & -(X_6-x_4)\Delta_4^6 & -(Y_6-y_4)\Delta_4^6 & -(Z_6-z_4)\Delta_4^6 \end{vmatrix} \tag{28}$$

(Check with the expression [E17] of Appendix E).

In terms of fundamental invariants (see Appendix F) each of these determinants may be expressed as follows:

$$|\Delta|_{j_1 j_2 \cdots j_6}=\sum_{\rho_1(\rho_2)}(-1)^{\sigma_1}(-1)^{\sigma_2}\begin{vmatrix} x_2 & 1 \\ X_{j1} & 1 \end{vmatrix}\begin{vmatrix} x_3 & y_3 & 1 \\ X_{j2} & Y_{j2} & 1 \\ X_{j3} & Y_{j3} & 1 \end{vmatrix}\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ X_{j4} & Y_{j4} & Z_{j4} & 1 \\ X_{j5} & Y_{j5} & Z_{j5} & 1 \\ X_{j6} & Y_{j6} & Z_{j6} & 1 \end{vmatrix}\Delta_4^{j6}\Delta_4^{j5}\Delta_4^{j4}\Delta_3^{j3}\Delta_3^{j2}\Delta_2^{j1} \tag{29}$$

This expression is obtained by two imbedded steps of Laplace expansion, namely:

1st step-expansion according to the last three columns, and

2nd step-expansion of the cofactor of each term in the first step according to its last two columns.

σ_1 and σ_2 result from the two imbedded Laplace expansions respectively. The summation is a double one. The outer summation being taken over the ρ_1 selections of row indices taken three at a time, and the inner one over the ρ_2 selections of the three complementary row indices for each of the ρ_1 selections taken two at a time. Thus summation (29) extends over

$$\rho_1\rho_2 = C_3^6\, C_2^3 = \frac{6!}{3!3!}\,\frac{3!}{2!2!} = 60 \text{ terms}$$

These determinants are relative invariants (see Appendix C) and when equated to zero, a hypersurface results which is an absolute invariant , for multiplication by a factor would have no significance in its equation (see Appendix F). One observes that each of these determinants is a functional of twenty-four variables. Since all the variables are coordinates of points in the geometric space, one may analyze, so to speak, that hypersurface into 3-dimensional loci. Each of these loci is a part of the hypersurface and all together constitute the hypersurface. To carry out that analysis a systematic procedure which fits the geometric characteristics of the problem will be beneficial. Thus consider all the variables involved falling in two groups: (A) the coordinates of the satellite-position points, and (B) the coordinates of the station-position points. Then two categories of loci will be considered: (1) the loci for the satellite-position points, and (2) the loci for the station-position points.

(1) Determinantal Loci for the Satellite-Position Points

Consider the equation

$$|\Delta|_{j_1, j_2, \ldots, j_6} = 0 \tag{30}$$

where $j_1, j_2, \ldots, j_6$ is a selection of six indices from $\{1,2,\ldots,m\}$. If (30) is satisfied for any six satellite-position points, then the rank of the jacobian matrix (table 3) is less than six and consequently one has a determinantal locus the analytic representation of which is (30). This is a hypersurface in 24-dimensional space. However, one may obtain 3-dimensional loci on this hypersurface by keeping the coordinates of all the points but one constant. One observes from (29) that the left-hand side of (30) is a functional of

i) second degree with respect to the coordinates X_j, Y_j, Z_j of any of the satellite-position points,

ii) sixth degree with respect to the coordinate x_2,

iii) fifth degree with respect to the coordinates x_3, y_3, and

iv) fourth degree with respect to the coordinates x_4, y_4, z_4.

Thus the three dimensional loci on that hypersurface for different points are generally different. Here the three dimensional loci on the hypersurface (30), for a satellite-position point will be considered.

Consider, without loss of generality, the hypersurface

$$|\Delta|_{1,\dots,6} = 0,$$

with all the coordinates involved being kept constant, except the coordinates of one of the satellite-position points, say X_6, Y_6, Z_6. Then $|\Delta|_{1,\dots,6}=0$ represents a second degree surface as a locus of the point $Q_6 \equiv (X_6, Y_6, Z_6)$. This surface passes through the four station-position points and the first five satellite-position points Q_1, Q_2, Q_3, Q_4, Q_5. As a matter of fact these nine points lie on that surface, for by plugging the coordinates of any of them in the place of X_6, Y_6, Z_6, the last row of $|\Delta|_{1,\dots,6}$ vanishes. Thus if for any selection $\{j_1,\dots,j_6\}$ out of $\{1,2,\dots,m\}$ the equation (30) is satisfied, which is equivalent to saying that all the satellite-position points lie on the second degree surface defined by the four station-position points and any five satellite-position points, then the rank of the jacobian matrix is less than six. Thus, the four station-position points and any five satellite-position points determine through (30) a determinantal locus in the three dimensional space (a second degree surface) for the rest points.

Now one proceeds to find out the linear (first degree) three dimensional determinantal loci.

Consider the expansion (29) of $|\Delta|_{j_1,\ldots,j_6}$. The value of this determinant is zero if each of its expansion-terms vanishes. By equating to zero each factor of the typical expansion-term in (29) one obtains the following determinantal loci.

(i) By putting $\begin{vmatrix} x_2 & 1 \\ X_{j_1} & 1 \end{vmatrix} = x_2 - X_j = 0$, $(j_1 = 1,2,\ldots,m)$, it is concluded that the plane perpendicular to the x-axis, passing through $P_2 \equiv (x_2, y_2, z_2)$ is a determinantal locus for the satellite-position points. In other words if all of them lie in that plane, all $|\Delta|_{j_1,\ldots,j_6}$ vanish, and then the rank of the jacobian matrix is less than six.

(ii) By equating the next factor to zero, i.e.,

$$\begin{vmatrix} x_3 & y_3 & 1 \\ X_{j_2} & Y_{j_2} & 1 \\ X_{j_3} & Y_{j_3} & 1 \end{vmatrix} = 0 \qquad (j_2, j_3 \text{ any two indices from } \{1,2,\ldots,m\}) \tag{31}$$

one has the collinearity condition of the projections on the (x,y)-plane of the three points $P_3 \equiv (x_3, y_3, z_3)$, $Q_{j2} \equiv (X_{j_2}, Y_{j_2}, Z_{j_2})$, $Q_{j_3} \equiv (X_{j_3}, Y_{j_3}, Z_{j_3})$.

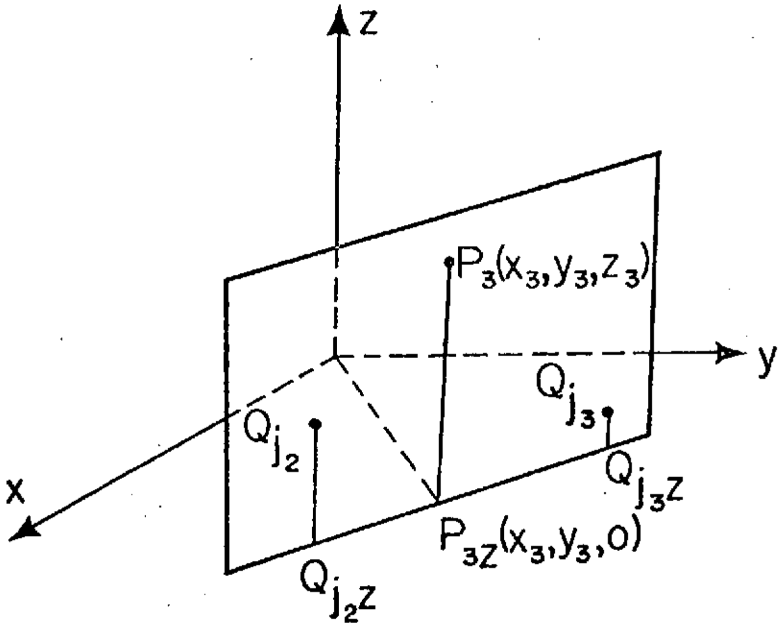

Figure 2

Thus if all the satellite-position points lie in a plane through $P_3 \equiv (x_3, y_3, z_3)$ and perpendicular to the (x,y)-plane, then relation (31) is satisfied for any j_2, j_3 from $\{1,2,\ldots,m\}$, and consequently each plane of the pencil of planes with axis the projection line of P_3 to the (x,y)-plane, is a determinantal locus for the satellite-position points, (see Figure 1).

(iii) Equating the third factor to zero one has

$$\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ X_{j_4} & Y_{j_4} & Z_{j_4} & 1 \\ X_{j_5} & Y_{j_5} & Z_{j_5} & 1 \\ X_{j_6} & Y_{j_6} & Z_{j_6} & 1 \end{vmatrix} = 0 \qquad \text{(for any } j_4, j_5, j_6 \text{ from } \{1,2,\ldots,m\}) \tag{32}$$

This is a condition for four points in 3-dimensional space to be linearly dependent, that is for the points,

$P_4 \equiv (x_4,y_4,z_4)$, $Q_{j_4} \equiv (X_{j_4},Y_{j_4},Z_{j_4})$, $Q_{j_5} \equiv (X_{j_5},Y_{j_5},Z_{j_5})$, $Q_{j_6} \equiv (X_{j_6},Y_{j_6},Z_{j_6})$ to be coplanar. Therefore if all the satellite-position points lie in a plane through the point $P_4 \equiv (x_4,y_4,z_4)$ relation (32) is satisfied for any triple j_4,j_5,j_6 from $1,2,\ldots,m$ and consequently each plane of the sheaf of planes through the point P_4 is a determinantal locus for the satellite-position points.

(iv) By equating either of the next three factors to zero, take

$$\Delta_4^{j_4} = 0,$$

one has the coplanarity condition of the points

$P_1 \equiv (x_1,y_1,z_1)$, $P_2 \equiv (x_2,y_2,z_2)$, $P_3 \equiv (x_3,y_3,z_3)$, $Q_{j_4} \equiv (X_{j_4},Y_{j_4},Z_{j_4})$. Therefore the plane of the station-position points P_1 , P_2, P_3 is a determinantal locus for the set of the satellite-position points. Similarly from

$$\Delta_3^{j_2} = 0 \quad \text{or} \quad \Delta_3^{j_3} = 0$$

one concludes that the plane (P_1,P_2,P_4) is a determinantal locus for the set of the satellite-position points and from

$$\Delta_2^{j_1} = 0,$$

that this is the case for the plane (P_1,P_3,P_4). Notice that the last two loci (i.e. the planes (P_1,P_2,P_4) and (P_1,P_3,P_4)) fall in the above considered case (iii). These are the determinantal loci for the satellite-position points independently from the relative position of the stations.

Remark. In addition to the above determinantal loci one could quote those which are degenerate cases of the above. These determinantal loci lower further the rank of the jacobian matrix, when it has already been lowered from six at least to five by one of the above quoted determinantal loci. For example, if all the satellite-position points are on a straight line, this is obviously a determinantal locus that is included in (32) for then there exists always a plane through P_4 and the straight line carrier of the satellite-position points. Such a loci which lower the rank further, once it has already been lowered from six, are trivial and result in an obvious way, being of no interest in this context.

(2) Determinantal Loci for the Station-Position Points

One is interested here for the determinantal loci for the station-position points independently from the satellite-position points. Since there are only four stations the only loci are the straight line determined by any two station-position points and the plane determined by any three of them. In the following the plane determined by any three station-position points is considered first. Suppose that the four station-position points are coplanar. Consider the determinants

$$\Delta_1^j,\ \Delta_2^j,\ \Delta_3^j,\ \Delta_4^j,$$

expanded according to the first row, i.e.

$$\Delta_1^j = \alpha_1 X_j + \beta_1 Y_j + \gamma_1 Z_j + \delta_1 \qquad (33)_1$$

$$\Delta_2^j = \alpha_2 X_j + \beta_2 Y_j + \gamma_2 Z_j + \delta_2 \qquad (33)_2$$

$$\Delta_3^j = \alpha_3 X_j + \beta_3 Y_j + \gamma_3 Z_j + \delta_3 \qquad (33)_3$$

$$\Delta_4^j = \alpha_4 X_j + \beta_4 Y_j + \gamma_4 Z_j + \delta_4 \qquad (33)_4$$

where α_i, β_i, γ_i, δ_i, $(i=1,2,3,4)$ are functions of the station coordinates only, (see relations 27).

By putting

$$\rho_i = \varepsilon(\alpha_i^2 + \beta_i^2 + \gamma_i^2)^{1/2}, \quad (\varepsilon \equiv \text{sign of } \Delta_i^j) \tag{34}$$

one has

$$\frac{\Delta_1^j}{\rho_1} = \frac{\Delta_2^j}{\rho_2} = \frac{\Delta_3^j}{\rho_3} = \frac{\Delta_4^j}{\rho_4} = d \tag{35}$$

where d is the distance of $Q_j \equiv (X_j, Y_j, Z_j)$ from the plane of the four station-position points. Indeed

$$\frac{\Delta_i^j}{\rho_i}, (i = 1,2,3,4)$$

is the expression of the distance of the point $Q_j \equiv (X_j, Y_j, Z_j)$ from the plane of three of the four station-position points, and if all four are coplanar (35) follows.

From (35) and relations (E12) and (E13) of Appendix E one obtains:

$$0 = \Delta_1^j - \Delta_2^j + \Delta_3^j - \Delta_4^j = d(\rho_1 - \rho_2 + \rho_3 - \rho_4) \Rightarrow$$

$$\rho_1 + \rho_3 = \rho_2 + \rho_4, \quad (d \neq 0). \tag{36}$$

Now consider the typical row of the jacobian matrix (Table 3) and effect the following rank-equivalent operations, in the sequence they are given

(i) Divide each row by the corresponding Δ_4^j (*)

(ii) Subtract from x_2-column, ρ_2/ρ_4 times the x_4-column.

(iii) Add from x_3-column, ρ_3/ρ_4 times the x_4-column.

(iv) Add from y_3-column, ρ_3/ρ_4 times the y_4-column.

(v) Change the signs of the columns x_3, y_3, x_4, y_4, z_4.

Then the rank-equivalent jacobian matrix is:

*One may divide by any of Δ_i^j (i = 1,2,3,4)

x_2	x_3	y_3	x_4	y_4	z_4
$(x_2-x_4)\frac{\rho_2}{\rho_4}$	$(x_4-x_3)\frac{\rho_3}{\rho_4}$	$(y_4-y_3)\frac{\rho_3}{\rho_4}$	X_j-x_4	Y_j-y_4	Z_j-z_4

$j=1,2,\ldots,m.$

Table 3. Rank-equivalent Condensed Jacobian Matrix for Four Coplanar Station-points.

One observes that each of the first three columns has the same entries throughout all the rows. Therefore the determinants of all 6x6 submatrices vanish and consequently the plane determined by any three of the four station-position points is a determinantal locus for the fourth one.

By considering the determinants Δ_i^j,(i=1,2,3,4) one concludes that if three station-position points are collinear one of them vanishes. Thus the straight line determined by any two station-position points is a determinantal locus for each of the other two separately.

Recapitulating the above results one has

1) Determinantal loci for the satellite-position points

1a) The four station-position points and any five satellite-position points determine through relation (30) a determinantal locus for the rest of the satellite-position points.

1b) The plane through $P_2 \equiv (x_2, y_2, z_2)$ and perpendicular to the x-axis is a determinantal locus for the set of the satellite-position points.

1c) Each plane of the pencil of planes through $P_3 \equiv (x_3,y_3,z_3)$ and perpendicular to the (x,y)-plane is a determinantal locus for the set of the satellite-position points.

1d) Each plane of the sheaf of planes through $P_4 \equiv (x_4,y_4,z_4)$ is a determinantal locus for the set of the satellite-position points.

1e) Each of the planes $(P_1P_2P_3)$, $(P_1P_2P_4)$, and $(P_1P_3P_4)$ is a determinantal locus for the set of the satellite-position points. Although $(P_1P_2P_4)$ and $(P_1P_3P_4)$ are included in (1d) it seems appropriate to quote them along with $(P_1P_2P_3)$ which is not included in any of the above loci.

2) Determinantal loci for the station-position points

2a) The plane of any three of the four station-position points is a determinantal locus for the fourth one.

2b) The straight line through any two station-position points is a determinantal locus for each of the other two separately.

Remarks

i) No loci that lower the rank of the jacobian matrix further, once it has already been lowered from six, is considered above. These loci are trivial and are easily obtained as further degeneration of those already given. For example, if the satellite-position points lie on a straight line, then there is always a plane through that line and the station-position point $P_4 \equiv (x_4,y_4,z_4)$, and this is nothing but an obvious further degeneration of the above quoted case (1d).

ii) The above results correspond to the set of constraints x_1,y_1,z_1,y_2,z_2,z_3 = constant. However, for any other set of admissible constraints only the determinantal loci (1b), (1c), (1d), and (1e) will be changed. Having the symmetric condensed jacobian matrix of Table 1, one can carry out the above procedure for any other set of admissible constraints very easily.

1.4.3 The Case of Five Ground Stations

Let $P_1 \equiv (x_1,y_1,z_1)$, $P_2 \equiv (x_2,y_2,z_2)$, $P_3 \equiv (x_3,y_3,z_3)$ $P_4 \equiv (x_4,y_4,z_4)$, $P_5 \equiv (x_5,y_5,z_5)$ be the position points of five observing ground stations. Since the minimum number of participating stations at each event is four, there are

$$N = C_4^5 + C_5^5 = 5 + 1 = 6$$

possible basic observational patterns, (see section 1.1.4). As a matter of fact, they may observe either all together or in groups of four. Let $Q_j \equiv (X_j,Y_j,Z_j)$, $(j=1,2,\ldots,m)$ be a set of satellite-position points observed in some pattern by the above five stations. Then one has a set of simultaneous functional relationships of the form*

$$f_{ij} \equiv (X_j-x_i)^2 + (Y_j-y_i)^2 + (Z_j-z_i)^2 = r_{ij}^2. \tag{37}$$

This system contains $3m+3x5 = 3(m+5)$ unknown variables, six of which are to be constrained, so that there actually are

$$\mu = 3(m+3) \quad \text{unknown variables.} \tag{38}$$

Thus the number of simultaneous equations (37) must be

$$\nu \geq \mu = 3(m+3) \tag{39}$$

and for a unique least squares solution

$$\text{rank } J_M = \mu = 3(m+3) \tag{40}$$

where J_M is the jacobian matrix of f_{ij} with respect to X_j,Y_j,Z_j, x_i,y_i,z_i, $(j=1,2,\ldots,m,\ i=1,2,3,4,5)$. If

$$\text{rank } J_M < \mu = 3(m+3) \tag{41}$$

*In this case for each j the index i takes on at least four out of the five values 1,2,3,4,5.

one has the determinantal loci of the configuration. In order to find out the determinantal loci one must search for the zeroes of the determinants of all $\mu x \mu$ submatrices of the jacobian matrix J_M.

The expression of the jacobian matrix J_M varies with the observational pattern; however, it is not difficult to write down a general expression for the jacobian matrix, in the sense that it comprises the jacobian matrices of all possible observational patterns. In order to form that general expression, consider all possible station-quadruples, i.e.,

$$
\begin{aligned}
0_5 &\equiv (1,2,3,4)\\
0_4 &\equiv (1,2,3,5)\\
0_3 &\equiv (1,2,4,5)\\
0_2 &\equiv (1,3,4,5)\\
0_1 &\equiv (2,3,4,5).
\end{aligned}
\tag{42}
$$

Each satellite-position point is observed either by all five stations or by one of the above station-quadruples. It turns out that one can condense the jacobian matrix as in the case of four stations. The only new point in the procedure of getting the condensed jacobian matrix in this case, is that about the satellite-position points observed by all five stations. One verifies very easily by pivotal condensation that the condensed jacobian matrix corresponding to the case when all five stations observe simultaneously a set of satellite-position points, is formed by the condensed jacobian matrices of any two station-quadruples, say $0_5 \equiv (1,2,3,4)$ and $0_4 \equiv (1,2,3,5)$ placed rowwise one after the other. It is understood, of course, that the condensed jacobian matrices of these two station-quadruples correspond to the very same set of satellite=position points. Hence it became obvious the way of forming the general expression for the jacobian matrix. Let $\{j_5\}$, $\{j_4\}$, $\{j_3\}$, $\{j_2\}$ $\{j_1\}$ be the index-sets of the satellite-position points observed by the station-quadruples $0_5, 0_4, 0_3, 0_2, 0_1$ respectively, two and only two of which (no matter which ones) comprise* the satellite-position points observed by all five stations. By getting the condensed jacobian matrices of all station-quadruples separately, and placing them rowwise one after the other (regardless order) one forms the general expression for the condensed jacobian matrix for five stations. That is given in Table 4 where the symbol

$$\Delta^j_{ik}; \; i,k = 1,2,3,4,5,$$

is similar to Δ^j_i that was used previously in the case of four stations, (see relations (27). Namely it stands for the determinant of 4-th order whose top row is occupied by the affine coordinates of the satellite-position point Q_j (indicated by the upper index) and the rest three by the affine coordinates of three out of the five station-position points (in order of increasing index), excluding the stations P_i and P_k (indicated by the lower indices). For example

$$\Delta^j_{23} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_4 & y_4 & z_4 & 1 \\ x_5 & y_5 & z_5 & 1 \end{vmatrix} \tag{43}$$

The order of the rows is conventional, though. Some other order could be stipulated.

*Each of those two station-quadruples comprises the satellite-position points observed by all five stations, in addition to those which are observed by that station-quadruple alone.

Quad.	x_1	y_1	z_1	x_2	y_2	z_2	x_3	y_3	z_3	x_4	y_4	z_4	x_5	y_5	z_5
1,2,3,4	$(X_{j_5}-x_1)\Delta_{15}^{j_5}$	$(Y_{j_5}-y_1)\Delta_{15}^{j_5}$	$(Z_{j_5}-z_1)\Delta_{15}^{j_5}$	$-(X_{j_5}-x_2)\Delta_{25}^{j_5}$	$-(Y_{j_5}-y_2)\Delta_{25}^{j_5}$	$-(Z_{j_5}-z_2)\Delta_{25}^{j_5}$	$(X_{j_5}-x_3)\Delta_{35}^{j_5}$	$(Y_{j_5}-y_3)\Delta_{35}^{j_5}$	$(Z_{j_5}-z_3)\Delta_{35}^{j_5}$	$-(X_{j_5}-x_4)\Delta_{45}^{j_5}$	$-(Y_{j_5}-y_4)\Delta_{45}^{j_5}$	$-(Z_{j_5}-z_4)\Delta_{45}^{j_5}$	0	0	0
1,2,3,5	$(X_{j_4}-x_1)\Delta_{14}^{j_4}$	$(Y_{j_4}-y_1)\Delta_{14}^{j_4}$	$(Z_{j_4}-z_1)\Delta_{14}^{j_4}$	$-(X_{j_4}-x_2)\Delta_{24}^{j_4}$	$-(Y_{j_4}-y_2)\Delta_{24}^{j_4}$	$-(Z_{j_4}-z_2)\Delta_{24}^{j_4}$	$(X_{j_4}-x_3)\Delta_{34}^{j_4}$	$(Y_{j_4}-y_3)\Delta_{34}^{j_4}$	$(Z_{j_4}-z_3)\Delta_{34}^{j_4}$	0	0	0	$-(X_{j_4}-x_5)\Delta_{54}^{j_4}$	$-(Y_{j_4}-y_5)\Delta_{54}^{j_4}$	$-(Z_{j_4}-z_5)\Delta_{54}^{j_4}$
1,2,4,5	$(X_{j_3}-x_1)\Delta_{13}^{j_3}$	$(Y_{j_3}-y_1)\Delta_{13}^{j_3}$	$(Z_{j_3}-z_1)\Delta_{13}^{j_3}$	$-(X_{j_3}-x_2)\Delta_{23}^{j_3}$	$-(Y_{j_3}-y_2)\Delta_{23}^{j_3}$	$-(Z_{j_3}-z_2)\Delta_{23}^{j_3}$	0	0	0	$(X_{j_3}-x_4)\Delta_{43}^{j_3}$	$(Y_{j_3}-y_4)\Delta_{43}^{j_3}$	$(Z_{j_3}-z_4)\Delta_{43}^{j_3}$	$+(X_{j_3}-x_5)\Delta_{53}^{j_3}$	$+(Y_{j_3}-y_5)\Delta_{53}^{j_3}$	$+(Z_{j_3}-z_5)\Delta_{53}^{j_3}$
1,3,4,5	$(X_{j_2}-x_1)\Delta_{12}^{j_2}$	$(Y_{j_2}-y_1)\Delta_{12}^{j_2}$	$(Z_{j_2}-z_1)\Delta_{12}^{j_2}$	0	0	0	$-(X_{j_2}-x_3)\Delta_{32}^{j_2}$	$-(Y_{j_2}-y_3)\Delta_{32}^{j_2}$	$-(Z_{j_2}-z_3)\Delta_{32}^{j_2}$	$(X_{j_2}-x_4)\Delta_{42}^{j_2}$	$(Y_{j_2}-y_4)\Delta_{42}^{j_2}$	$(Z_{j_2}-z_4)\Delta_{42}^{j_2}$	$+(X_{j_2}-x_5)\Delta_{52}^{j_2}$	$+(Y_{j_2}=y_5)\Delta_{52}^{j_2}$	$+(Z_{j_2}-z_5)\Delta_{52}^{j_2}$
2,3,4,5	0	0	0	$(X_{j_1}-x_2)\Delta_{21}^{j_1}$	$(Y_{j_1}-y_2)\Delta_{21}^{j_1}$	$(Z_{j_1}-z_2)\Delta_{21}^{j_1}$	$-(X_{j_1}-x_3)\Delta_{31}^{j_1}$	$-(Y_{j_1}-y_3)\Delta_{31}^{j_1}$	$-(Z_{j_1}-z_3)\Delta_{31}^{j_1}$	$(X_{j_1}-x_4)\Delta_{41}^{j_1}$	$(Y_{j_1}-y_4)\Delta_{41}^{j_1}$	$(Z_{j_1}-z_4)\Delta_{41}^{j_1}$	$-(X_{j_1}-x_5)\Delta_{51}^{j_1}$	$-(Y_{j_1}-y_5)\Delta_{51}^{j_1}$	$-(Z_{j_1}-z_5)\Delta_{51}^{j_1}$

Table 4. The General Expression for the Condensed Jacobian Matrix for Five Stations in Range Networks.

Having formed the general expression for the condensed jacobian matrix, one proceeds to find out the determinantal loci. At first an admissible set of constraints is chosen. As in the case of four stations, take

$$x_1, y_1, z_1, y_2, z_2, z_3 = \text{constant}.$$

Since the columns of the jacobian matrix corresponding to the constrained coordinates are zero, the condensed jacobian matrix corresponding to the above chosen set of constraints results from Table 4 by simply leaving "out" the columns corresponding to those constrained coordinates. That matrix has 15-6 = 9 columns. Thus in order to find out the determinantal loci one has to search for the zeroes of the determinants of all 9x9 submatrices. Again, as in the case of four stations, the determinantal loci are distinguished in two categories:

i) loci for the satellite-position points, and

ii) loci for the station-position points.

The determinantal loci of each category are independent from one another, in the sense that each one may occur independently from the others.

1) <u>Determinantal loci for the satellite-position points</u>

For a systematic exposition the following two groups of patterns, (comprising all the patterns) will be considered separately.

(a) Two station-quadruples, say $O_5 = (1,2,3,4)$ and $O_5 \equiv (1,2,3,5)$ carry out all the observations. The theoretical treatment of this pattern includes that of the pattern when all five stations observe simultaneously all the satellite-position points.

(b) Three, four or all five station-quadruples carry out all the observations.

Remark. When all five stations observe simultaneously a certain set $\{Q_j\}$ of satellite-position points, one has the maximum number of equations he can have with the same set $\{Q_j\}$ and any possible observational pattern. The jacobian matrix corresponding to any observational pattern can be obtained from that of the pattern when all stations observe simultaneously by omitting certain equations. For example, suppose that the point-set

$$\{Q_j\},\ j\varepsilon\{j\}\equiv\{1,2,3,\ldots,m\}$$

has been observed by the quadruples O_5, O_4 and O_3, so that O_5 has observed the subset

$$\{Q_{j_5}\},\ j_5\varepsilon\{j_5\}\subset\{j\},$$

O_4 the subset $\{Q_{j_4}\},\ j_4\varepsilon\{j_4\}\subset\{j\}$,

and O_3 the subset $\{Q_{j_3}\},\ j_3\varepsilon\{j_3\}\subset\{j\}$,

where

$$\{j_3\}\cup\{j_4\}\cup\{j_5\}=\{j\};\ \{j_3\}\cap\{j_4\}=\phi;\ \{j_4\}\cap\{j_5\}=\phi;\ \{j_5\}\cap\{j_3\}=\phi.$$

The jacobian matrix of this pattern, results from the jacobian matrix of the pattern when all five stations observe simultaneously all the points in $\{Q_j\}$, by omitting the rows corresponding to the observations of stations P_3, P_4 and P_5 toward the point-sets $\{Q_{j_3}\}$, $\{Q_{j_4}\}$, and $\{Q_{j_5}\}$ respectively. Therefore, by considering the jacobian matrix as if all stations observe simultaneously, no matter what the actual observational pattern is, and checking its rank, if it is less than nine then this is the case for the jacobian matrix of the actual pattern as well; however if it is equal to nine, the rank of the jacobian matrix of the actual pattern may or may not be equal to nine, and one must examine the rank of the actual jacobian matrix.

From this remark becomes obvious why the pattern of two station-quadruples carrying out all the observations is so important to be treated separately, in searching for the determinantal loci in the case of five stations.

(a) Two station-quadruples carry out all the observations

Let $0_5 \equiv (1,2,3,4)$ and $0_4 \equiv (1,2,3,5)$ be the two station-quadruples. The condensed jacobian matrix of this case is given by the first two row-blocks of Table 4,

$$\{Q_{j_5}\} \equiv \{(X_{j_5}, Y_{j_5}, Z_{j_5})\},\ j_5 \varepsilon \{j_5\},$$

is the set of the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ and

$$\{Q_{j_4}\} \equiv \{(X_{j_4}, Y_{j_4}, Z_{j_4})\},\ j_4 \varepsilon \{j_4\},$$

is the set of the satellite-position points observed by $0_4 \equiv (1,2,3,5)$. Note that for

$$\{Q_{j_5}\} \equiv \{Q_{j_4}\} \equiv \{Q_j\},\ (*) \qquad (44)$$

one has the case when all five stations observe simultaneously all the satellite-position points, while for

$$\{Q_{j_5}\} \cap \{Q_{j_4}\} = \phi, \qquad (45)$$

the case when there is no event at which stations P_4 and P_5 enter together. In the case when some satellite-position points are observed by all five stations, while the rest by either of the two station-

*It is assumed that $\{Q_{j_5}\} \cup \{Q_{j_4}\} = \{Q_j\}$ is the set of all the satellite-position points available.

	x_2	x_3	y_3	x_4	y_4	z_4	x_5	y_5	z_5
1, 2, 3, 4	$-(X_{j_{51}}-x_2)\Delta^{j_{51}}_{25}$	$(X_{j_{51}}-x_3)\Delta^{j_{51}}_{35}$	$(Y_{j_{51}}-y_3)\Delta^{j_{51}}_{35}$	$-(X_{j_{51}}-x_4)\Delta^{j_{51}}_{45}$	$-(Y_{j_{51}}-y_4)\Delta^{j_{51}}_{45}$	$-(Z_{j_{51}}-z_4)\Delta^{j_{51}}_{45}$	0	0	0
	...	...	...	...	...	...	...	...	...
	$-(X_{j_{5i}}-x_2)\Delta^{j_{5i}}_{25}$	$(X_{j_{5i}}-x_3)\Delta^{j_{5i}}_{35}$	$(Y_{j_{5i}}-y_3)\Delta^{j_{5i}}_{35}$	$-(X_{j_{5i}}-x_4)\Delta^{j_{5i}}_{45}$	$-(Y_{j_{5i}}-y_4)\Delta^{j_{5i}}_{45}$	$-(Z_{j_{5i}}-z_4)\Delta^{j_{5i}}_{45}$	0	0	0
	...	...	...	...	...	...	...	...	...
	x_2	x_3	y_3	x_4	y_4	z_4	x_5	y_5	z_5
1, 2, 3, 5	$-(X_{j_{41}}-x_2)\Delta^{j_{41}}_{24}$	$(X_{j_{41}}-x_3)\Delta^{j_{41}}_{34}$	$(Y_{j_{41}}-y_3)\Delta^{j_{41}}_{34}$	0	0	0	$-(X_{j_{41}}-x_5)\Delta^{j_{41}}_{54}$	$-(Y_{j_{41}}-y_5)\Delta^{j_{41}}_{54}$	$-(Z_{j_{41}}-z_5)\Delta^{j_{41}}_{54}$
	...	...	...	...	...	...	...	...	...
	$-(X_{j_{4i}}-x_2)\Delta^{j_{4i}}_{24}$	$(X_{j_{4i}}-x_3)\Delta^{j_{4i}}_{34}$	$(Y_{j_{4i}}-y_3)\Delta^{j_{4i}}_{34}$	0	0	0	$-(X_{j_{4i}}-x_5)\Delta^{4i}_{54}$	$-(Y_{j_{4i}}-y_5)\Delta^{4i}_{54}$	$-(Z_{j_{4i}}-z_5)\Delta^{4i}_{54}$
	...	...	...		...	...	...	...	...

Table 5. The Condensed Jacobian Matrix of the Case When the Station-Quadruples O_5 and O_4 carry out all the observations, under the Set of Admissible Constraints x_1, y_1, z_1, y_2, z_2, z_3 = Constant.

quadruples alone, neither (44) nor (45) is true.

Considering the above adopted set of admissible constraints, the condensed jacobian matrix of the present case is given by Table 5 (on preceding page), that is nothing else but the first two row blocks of Table 4 after omitting the columns of the constrained coordinates x_1, y_1, z_1, y_2, z_2, z_3.

For the determinantal loci one has to search for the zeroes of the determinants of all 9x9 submatrices. In order these determinants not to be identically equal to zero, each one of them must include at least three rows from each station-quadruple or that is equivalent at most six rows from each station-quadruple.

Since the rank of the jacobian matrix is invariant under affine trnasformation (see Appendix C) and the same is true for the determinants of its submatrices, one can express each of the determinants of the 9x9 submatrices as polynomials in the fundamental invariants (see Appendix F). This is done by Laplace expansion in the same way that was done for the case of four stations. Thus the determinant, $|\Delta(0_5,0_4)|$, of the submatrix corresponding to a certain selection $k_1,k_2,\ldots,k_9$ of satellite-point-indices (satisfying the following conditions (47) is given by the following messy, but nevertheless useful, summation formula.

$$|\Delta(0_5,0_4)| = \sum_{\rho_1(\rho_2(\rho_3))} (-1)^{\sigma_1+\sigma_2+\sigma_3} \begin{vmatrix} x_2 & 1 \\ X_{k_1} & 1 \end{vmatrix} \begin{vmatrix} x_3 & y_3 & 1 \\ X_{k_2} & Y_{k_2} & 1 \\ X_{k_3} & Y_{k_3} & 1 \end{vmatrix} \mathfrak{D}_4^{k_4,k_5,k_6} \mathfrak{D}_5^{k_7,k_8,k_9} \Delta_{45}^{k_4} \Delta_{45}^{k_5} \Delta_{45}^{k_6} \Delta_{54}^{k_7} \Delta_{54}^{k_8} \Delta_{54}^{k_9}$$
$$\times \; \Delta_{2\lambda}^{k_1} \quad \Delta_{3\mu}^{k_2} \quad \Delta_{3\nu}^{k_3} \quad , \tag{46}$$

where

$$\begin{aligned} k_1 &\in \{j_5\} \cup \{j_4\} , \\ k_2,\ k_2 &\in \{j_5\} \cup \{j_4\} , \\ k_4,\ k_5,\ k_6 &\in \{j_5\} , \\ k_7,\ k_8,\ k_9 &\in \{j_4\} , \end{aligned} \tag{47}$$

$$\lambda,\mu,\nu = \begin{cases} 4, & \text{if } k_1,\ k_2,\ k_3 \in \{j_5\} \text{ , respectively,} \\ 5, & \text{if } k_1,\ k_2,\ k_3 \in \{j_4\} \text{ , respectively,} \end{cases} \tag{48}$$

$$\mathfrak{D}_4^{k\ ,k\ ,k} = \begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ X_{k_4} & Y_{k_4} & Z_{k_4} & 1 \\ X_{k_5} & Y_{k_5} & Z_{k_5} & 1 \\ X_{k_6} & Y_{k_6} & Z_{k_6} & 1 \end{vmatrix} , \quad \mathfrak{D}_5^{k_7,k_8,k_9} = \begin{vmatrix} x_5 & y_5 & z_5 & 1 \\ X_{k_7} & Y_{k_7} & Z_{k_7} & 1 \\ X_{k_8} & Y_{k_8} & Z_{k_8} & 1 \\ X_{k_9} & Y_{k_9} & Z_{k_9} & 1 \end{vmatrix} \tag{49}$$

By equating the determinant of a 9x9 submatrix to zero, i.e.,

$$|\Delta(0_5,0_4)| = 0 \tag{50}$$

one has a hypersurface with at least 24 and at most 36 variables. One may keep constant the coordinates of all the points involved but one, and consider (50) as a three dimensional locus of that point. However the form of (50) varies with the choice of the submatrix in such a way that searching for the three dimensional second order determinantal loci is not as simple a task as in the case of four stations. Before proceeding systematically, take for example the determinant of the following Table 6.

	x_2	x_3	y_3	x_4	y_4	z_4	x_5	y_5	z_5
$0_5 \equiv (1,2,3,4)$	$-(X_1-x_2)\Delta_{25}^{1}$	$(X_1-x_3)\Delta_{35}^{1}$	$(Y_1-y_3)\Delta_{35}^{1}$	$-(X_1-x_4)\Delta_{45}^{1}$	$-(Y_1-y_4)\Delta_{45}^{1}$	$-(Z_1-z_4)\Delta_{45}^{1}$	0	0	0
	$-(X_2-x_2)\Delta_{25}^{2}$	$(X_2-x_3)\Delta_{35}^{2}$	$(Y_2-y_3)\Delta_{35}^{2}$	$-(X_2-x_4)\Delta_{45}^{2}$	$-(Y_2-y_4)\Delta_{45}^{2}$	$-(Z_2-z_4)\Delta_{45}^{2}$	0	0	0
	$-(X_3-x_2)\Delta_{25}^{3}$	$(X_3-x_3)\Delta_{35}^{3}$	$(Y_3-y_3)\Delta_{35}^{3}$	$-(X_3-x_4)\Delta_{45}^{3}$	$-(Y_3-y_4)\Delta_{45}^{3}$	$-(Z_3-z_4)\Delta_{45}^{3}$	0	0	0
	$-(X_4-x_2)\Delta_{25}^{4}$	$(X_4-x_3)\Delta_{35}^{4}$	$(Y_4-y_3)\Delta_{35}^{4}$	$-(X_4-x_4)\Delta_{45}^{4}$	$-(Y_4-y_4)\Delta_{45}^{4}$	$-(Z_4-z_4)\Delta_{45}^{4}$	0	0	0
	$-(X_5-x_2)\Delta_{25}^{5}$	$(X_5-x_3)\Delta_{35}^{5}$	$(Y_5-y_3)\Delta_{35}^{5}$	$-(X_5-x_4)\Delta_{45}^{5}$	$-(Y_5-y_4)\Delta_{45}^{5}$	$-(Z_5-z_4)\Delta_{45}^{5}$	0	0	0
$0_4 \equiv (1,2,3,5)$	$-(X_6-x_2)\Delta_{24}^{6}$	$(X_6-x_3)\Delta_{34}^{6}$	$(Y_6-y_3)\Delta_{34}^{6}$	0	0	0	$-(X_6-x_5)\Delta_{54}^{6}$	$-(Y_6-y_5)\Delta_{54}^{6}$	$-(Z_6-z_5)\Delta_{54}^{6}$
	$-(X_7-x_2)\Delta_{24}^{7}$	$(X_7-x_3)\Delta_{34}^{7}$	$(Y_7-y_3)\Delta_{34}^{7}$	0	0	0	$-(X_7-x_5)\Delta_{54}^{7}$	$-(Y_7-y_5)\Delta_{54}^{7}$	$-(Z_7-z_5)\Delta_{54}^{7}$
	$-(X_8-x_2)\Delta_{24}^{8}$	$(X_8-x_3)\Delta_{34}^{8}$	$(Y_8-y_3)\Delta_{34}^{8}$	0	0	0	$-(X_8-x_5)\Delta_{54}^{8}$	$-(Y_8-y_5)\Delta_{54}^{8}$	$-(Z_8-z_5)\Delta_{54}^{8}$
	$-(X_9-x_2)\Delta_{24}^{9}$	$(X_9-x_3)\Delta_{34}^{9}$	$(Y_9-y_3)\Delta_{34}^{9}$	0	0	0	$-(X_9-x_5)\Delta_{54}^{9}$	$-(Y_9-y_5)\Delta_{54}^{9}$	$-(Z_9-z_5)\Delta_{54}^{9}$

Table 6. A 9x9 Submatrix of the Table 5.

The above table is a 9x9 submatrix of the jacobian matrix, which has five rows from the station-quadruple $O_4 \equiv (1,2,3,4)$ and four rows from O_5 (1,2,3,5). By considering its determinant as a functional with respect to X_5, Y_5, Z_5 (the coordinates of Q_5-one of the points observed by O_5) and equating it to zero, one obtains a second degree surface as the locus of the point $Q_5 \equiv (X_5, Y_5, Z_5)$, that passes obviously through Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3, P_4 and the origin (0,0,0). For each selection of the last four rows in the station-quadruple $O_4 \equiv (1,2,3,5)$ the determinant of the above table equated to zero, gives a second degree surface as the locus of $Q_5 \equiv (X_5, Y_5, Z_5)$ that passes through the same nine points. Provided these nine points do not constitute a singular configuration, they determine uniquely a second order surface passing through each one of them. Therefore the loci of $Q_5 \equiv (X_5, Y_5, Z_5)$ (second degree surfaces) corresponding to all possible selections of four rows in $O_4 \equiv (1,2,3,5)$ coincide with the second degree surface determined by the nine points Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3, P_4 and the origin, provided these points do not constitute a singular configuration. Note that instead of Q_5 one may consider any of the satellite-position points observed by $O_5 \equiv (1,2,3,4)$.

Similarly, by considering the determinant of the above Table 6 as a functional with respect to X_9, Y_9, Z_9 (the coordinates of Q_9-one of the points observed by $O_4 \equiv (1,2,3,5)$) and equating it to zero, one obtains a second degree surface as the locus of $Q_9 \equiv (X_9, Y_9, Z_9)$, that obviously passes through the points Q_6, Q_7, Q_8, P_1, P_2, P_3, P_5 and the origin (0,0,0). For each selection of the first five rows in the station-quadruple $O_5 \equiv (1,2,3,4)$ the determinant of the above table being considered as a functional of X_9, Y_9, Z_9 and equated to zero, gives a second degree surface as the locus of $Q_9 \equiv (X_9, Y_9, Z_9)$ that passes through the same eight points. But there are infinitely many second degree surfaces through eight points. Therefore the loci of $Q_9 \equiv (X_9, Y_9, Z_9)$ (second degree surfaces) corresponding to different selections of five rows in $0_5 \equiv (1,2,3,4)$ are generally different. However, instead of $Q_9 \equiv (X_9, Y_9, Z_9)$ one may consider any other of the satellite-position points observed by $0_4 \equiv (1,2,3,5)$. That is, the second degree surface that is determined by any selection of five rows in $0_5 \equiv (1,2,3,4)$, say those corresponding to Q_1, Q_2, Q_3, Q_4, Q_5, and any selection of three rows in $0_4 \equiv (1,2,3,5)$, say those corresponding to Q_6, Q_7, Q_8, passes through all the satellite-position points observed by the station-quadruple $0_4 \equiv (1,2,3,4)$, through the station-position points P_1, P_2, P_3, P_5, and the origin. Therefore, if the station-quadruple $0_4 \equiv (1,2,3,5)$, observes more than three satellite-position points, the very same second degree surface that is determined by the nine points: origin, P_1, P_2, P_3, P_5, and any four satellite-position points from the station-quadruple $0_4 \equiv (1,2,3,5)$, it is determined completely also by P_1, P_2, P_3, P_5, origin, and any three satellite-position points. Of course this is not the case in general with a second degree surface.

In order to find out the determinantal loci of second degree, one may proceed as follows.

As it was pointed out before, each 9x9 submatrix being considered for the determinantal loci must have at least three rows from each station-quadruple and at most six rows; otherwise its determinant is identically zero. Thus one considers all 9x9 submatrices of the jacobian matrix falling in four categories:

(α) The submatrices which have six rows from the station-quadruple $0_5 \equiv (1,2,3,4)$ and three rows from $0_4 \equiv (1,2,3,5)$,

(β) The submatrices which have five rows from $0_5 \equiv (1,2,3,4)$ and four rows from $0_4 \equiv (1,2,3,5)$,

(λ) The submatrices which have four rows from $0_5 \equiv (1,2,3,4)$ and five rows from $0_4 \equiv (1,2,3,5)$, and

(δ) The submatrices which have three rows from $0_5 \equiv (1,2,3,4)$ and six rows from $0_4 \equiv (1,2,3,5)$.

The determinantal loci for the satellite-position points observed by the station-quadruple $0_5 \equiv (1,2,3,4)$ will be considered below.

The determinantal loci for the satellite-position points observed by the station-quadruple $0_4 \equiv (1,2,3,5)$ result in exactly the same way. One simply interchanges the roles of 0_5 and 0_4 in the following discussion for the determinantal loci of the points observed by $0_5 \equiv (1,2,3,4)$.

For the rank of the jacobian matrix to be less than nine the determinants of the submatrices of all the categories ((α), (β), (γ), (δ)) must vanish simultaneously. Thus the loci for the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ resulting from the four categories must coincide, since they refer to the same set of points.

<u>From the category (α)</u>: Let $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ be six points from the station-quadruple $0_5 \equiv (1,2,3,4)$ and Q_7, Q_8, Q_9 from $0_4 \equiv (1,2,3,5)$. By considering the determinant of the submatrix corresponding to these points as a functional with respect to the coordinates of one of the points observed by $0_5 \equiv (1,2,3,4)$, say $Q_6 \equiv (X_6, Y_6, Z_6)$ and equating it to zero, one has a second degree locus for $Q_6 \equiv (X_6, Y_6, Z_6)$, that passes

through the points Q_1,Q_2,Q_3,Q_4,Q_5, P_1,P_2,P_3,P_4 and the origin $\equiv(0,0,0)$. For each selection of three rows in $O_4\equiv(1,2,3,5)$ the locus of $Q_6\equiv(X_6,Y_6,Z_6)$ passes through the same points. Since this is a second degree surface, it can be determined uniquely by any nine of its points, provided they are not in singular configuration. Moreover instead of $Q_6\equiv(X_6,Y_6,Z_6)$ one may take any other point from those observed by $O_5\equiv(1,2,3,4)$. Thus the locus of the satellite-position points observed by $O_5\equiv(1,2,3,4)$, dictated by the category (α), is a second degree surface whose equation can be given either by equating to zero the determinant of a submatrix of the category (α) (being considered as a functional with respect to the coordinates of one of the points observed by $O_5\equiv(1,2,3,4)$), or by determining the second order surface which passes through the nine points: origin, P_1,P_2,P_3,P_4 and any four satellite-position points, observed by $O_5\equiv(1,2,3,4)$. Let this locus be denoted by L_5^3.

<u>From the category (β)</u>: This category was considered in the above given example. It was found that, for each selection of four rows in $O_4\equiv(1,2,3,5)$ the locus of the satellite-position points observed by $O_5\equiv(1,2,3,4)$ passes through the points: origin, P_1,P_2,P_3,P_4 and four from the points observed by $O_5\equiv(1,2,3,4)$. This locus is again a general second degree surface whose equation is given either by equating to zero the determinant of a submatrix of the category (β), (being considered as a functional with respect to the coordinates of one of the points observed by $O_5\equiv(1,2,3,4)$), or by determining the second order surface which passes through the nine points: origin, P_1,P_2,P_3,P_4 and any four points observed by $O_5\equiv(1,2,3,4)$. Obviously this locus

coincides with L_5^3 of category (α). Let it be denoted by L_5^4.

From category (γ): This category was also considered in that, there (second part of the example) the discussion was for the locus of the points observed by $0_4 \equiv (1,2,3,5)$ and not $0_4 \equiv (1,2,3,4)$. However, the results are similar. Thus by transferring those results here one has that, for each selection of five rows in $0_4 \equiv (1,2,3,5)$ the locus of the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ passes through the points: origin, P_1, P_2, P_3, P_4 and any three points observed by $0_5 \equiv (1,2,3,4)$. But eight points do not determine a general second degree surface, i.e., there are infinitely many general second degree surfaces passing through eight points. However, all the loci corresponding to different selections of five rows from $0_4 \equiv (1,2,3,5)$ refer to the very same set of points that is the points observed by $0_5 \equiv (1,2,3,4)$. Therefore they must coincide with L_5^3 and L_5^4, in case of a determinantal locus. That implies that the locus of the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ is not a general second degree surface, but it is such that any eight of its points determine it completely. The equation of this locus is given by equating to zero the determinant of a submatrix of the category (γ), (being considered as a functional with respect to the coordinates of one of the points observed by $0_5 \equiv (1,2,3,4)$). For the existence of such a locus take the equation of L_5^3 ($\equiv L_5^4$) and verify that the same locus L_5^3 is determined completely by any eight of its points. Let such a locus be denoted by L_5^5.

From the category (δ): Let Q_1, Q_2, Q_3 be three points observed by $0_5 \equiv (1,2,3,4)$ and $Q_4, Q_5, Q_6, Q_7, Q_8, Q_9$ six points observed by $0_4 \equiv (1,2,3,5)$.

By considering the determinant of the submatrix corresponding to these points as a functional with respect to the coordinates of one of the points observed by $0_5 \equiv (1,2,3,4)$, say $Q_3 \equiv (X_3,Y_3,Z_3)$, and equating it to zero, one obtains a second degree surface as the locus of $Q_3 \equiv (X_3,Y_3,Z_3)$, that passes through the points Q_1,Q_2,P_1,P_2,P_3,P_4 and the origin $\equiv (0,0,0)$. For each selection of six rows in $0_4 \equiv (1,2,3,5)$ the locus of $Q_3 \equiv (X_3,Y_3,Z_3)$ passes through the same seven points. Instead of Q_3 one may consider any point observed by $0_5 \equiv (1,2,3,4)$. Thus the locus of the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ in this category, passes through the points: origin, P_1,P_2,P_3,P_4 and any two from the points observed by $0_5 \equiv (1,2,3,4)$. The equation of this locus is taken by equating to zero the determinant of a submatrix of this category (δ), (being considered as a functional with respect to the coordinates of one of the points observed by $0_5 \equiv (1,2,3,4)$). This locus must coincide with the loci L_5^3, L_5^4, and L_5^5 of the previous categories, in case of a determinantal locus. That implies that the locus of the points observed by $0_5 \equiv (1,2,3,4)$ is not a general second degree surface but it is such that any seven of its points determine it completely. Let such a locus be denoted by L_5^6. For the existence of such a locus take the equation of L_5^3 ($\equiv L_5^4$) and verify that the same locus L_5^3 is determined completely by any six of its points.

The same discussion may be repeated for the determinantal loci of the satellite-position points observed by $0_4 \equiv (1,2,3,5)$, to find the completely analogous loci L_4^3, L_4^4, L_4^5, L_4^6. In the general case, when both station-quadruples $0_5 \equiv (1,2,3,4)$ and $0_4 \equiv (1,2,3,5)$ observe six or more points each, the determinantal locus is L_5^6 for the points observed by

$0_5 \equiv (1,2,3,4)$ and L_4^6 for the points observed by $0_4 \equiv (1,2,3,5)$. It is understood that these loci, L_5^6 and L_4^6, go together. That means that for the rank of the jacobian matrix to be less than nine, both the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ must lie on L_5^6, and the satellite-position points observed by $0_4 \equiv (1,2,3,5)$ must lie on L_4^6. It is possible L_5^6 and L_4^6 to coincide. That happens whenever they have seven* points in common; and since they do have in common the points P_1, P_2, P_3, P and the origin, at least three satellite-position points observed by all five stations are sufficient for the loci L_5^6 and L_4^6 to coincide. Besides the general case when each quadruple observes six or more points, one has some specific cases, which are those when one of the quadruples observes either only three or only four or only five points. Obviously, if one of the quadruples observes only three points, one has to consider only the category (α), if it observes only four points, one has to consider only the categories, (α) and (β), and if it observes only five points, one has to consider the categories (α), (β) and (γ).

After the second degree determinantal loci, one proceeds to consider the linear ones. They may be taken from the expression (46) by equating to zero each factor of the typical term of the summation. Thus one has:

(i) From

$$\begin{vmatrix} x_2 & 1 \\ X_{k_1} & 1 \end{vmatrix} = 0, \quad (k_1 \varepsilon \{j_5\} \cup \{j_4\}) \tag{51}$$

*Seven points, because as it was said above (category (δ)) L_5^6 (and its counterlocus L_4^6) is not a <u>general</u> second degree surface, but it is such that any seven of its points determine it completely.

it is concluded that if all the satellite-position points, (observed by both station quadruples), lie in a plane perpendicular to the x-axis and passing through the point $P_2 \equiv (x_2, y_2, z_2)$, the first column of all 9x9 submatrices i a zero column. Therefore that plane is a determinantal locus for <u>all</u> the points.

(ii) From

$$\begin{vmatrix} x_3 & y_3 & 1 \\ X_{k_2} & Y_{k_2} & 1 \\ X_{k_3} & Y_{k_3} & 1 \end{vmatrix} = 0, \quad (k_2, k_3 \varepsilon \{j_5\} \cup \{j_4\}) \tag{52}$$

it is concluded that if all the satellite-position points lie in a plane through the point $P_3 \equiv (x_3, y_3, z_3)$ and perpendicular to the (xy)-plane, the determinants of all 9x9 submatrices are zero. Therefore each plane of the pencil of planes with axis the projection line of P_3 to the (x,y)-plane is a determinantal locus for <u>all</u> the points.

(iii) From

$$\mathcal{D}_4^{k_4, k_5, k_6} = \begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ X_{k_4} & Y_{k_4} & Z_{k_4} & 1 \\ X_{k_5} & Y_{k_5} & Z_{k_5} & 1 \\ X_{k_6} & Y_{k_6} & Z_{k_6} & 1 \end{vmatrix}, \quad (k_4, k_5, k_6 \varepsilon \{j_5\}) \tag{53}$$

it is concluded that, if the satellite-position points observed by the station-quadruple $O_5 \equiv (1,2,3,4)$ lie in a plane through the point $P_4 \equiv (x_4, y_4, z_4)$, then the determinants of all 9x9 submatrices are zero. Therefore, each plane of the sheaf of planes through the point P_4 is a determinantal locus for the satellite-position points observed by $O_5 \equiv (1,2,3,4)$.

(iv) From

$$\mathfrak{D}_5^{k_7,k_8,k_9} = \begin{vmatrix} x_5 & y_5 & z_5 & 1 \\ X_{k_7} & Y_{k_7} & Z_{k_7} & 1 \\ X_{k_8} & Y_{k_8} & Z_{k_8} & 1 \\ X_{k_9} & Y_{k_9} & Z_{k_9} & 1 \end{vmatrix}, \quad (k_7,k_8,k_9 \varepsilon \{j_4\}) \tag{54}$$

it is concluded that, if the satellite-position points, observed by the station-quadruple $O_4 \equiv (1,2,3,5)$, lie in a plane through the point $P_5 \equiv (x_5, y_5, z_5)$, then the determinants of all 9x9 submatrices are zero. Therefore each plane of the sheaf of planes through the point P_5 is a determinantal locus for the satellite-position points observed by the station-quadruple $O_4 \equiv (1,2,3,5)$.

(v) From either one of the following three relations

$$\Delta_{45}^{k4} = 0, \; \Delta_{45}^{k5} = 0, \; \Delta_{45}^{k6} = 0, (k4, k5, k6 \varepsilon \{j5\}), \tag{55}$$

it is concluded that the plane (P_1, P_2, P_3) is a determinantal locus for the set of the satellite-position points observed by $O_5 \equiv (1,2,3,4)$.

(vi) From either one of the following three relations

$$\Delta_{54}^{k7} = 0, \; \Delta_{54}^{k8} = 0, \; \Delta_{54}^{k9} = 0, (k7, k8, k9 \varepsilon \{j4\}), \tag{56}$$

it is concluded that the plane (P_1, P_2, P_3) is also a determinantal locus for the satellite-position points observed by $O_4 \equiv (1,2,3,5)$.

<u>Note:</u> In the last two cases, (v) and (vi), it is observed that, one may have the same determinantal locus for different sets of satellite-position points.

(vii) From

$$\Delta_{2\lambda}^{k_1} = 0, \; k_1 \varepsilon \{j_5\} \cup \{j_4\}, \; \lambda = \begin{cases} 4, & \text{if } k_1 \varepsilon \{j_5\} \\ 5, & \text{if } k_1 \varepsilon \{j_4\} \end{cases} \tag{57}$$

it is concluded that, if the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ lie in the plane (P_1, P_3, P_5) and simultaneously the satellite-position points observed by $0_4 \equiv (1,2,3,5)$ lie in the (P_1, P_3, P_4), then the determinants of all 9x9 submatrices are zero. Therefore the planes (P_1, P_3, P_5) and (P_1, P_3, P_4) are simultaneously determinantal loci for the satellite-position points observed by $0_5 \equiv (1,2,3,4)$ and $0_4 \equiv (1,2,3,5)$ respectively.

(viii) From either one of the following two relations

$$\Delta_{3\mu}^{k2} = 0, \quad \Delta_{3\nu}^{k3} = 0, \qquad k_2, k_3 \varepsilon \{j_5\} \cup \{j_4\}, \quad \mu,\nu = \begin{matrix} 4, \text{ if } k_2, k_3 \varepsilon \{j_5\} \text{ respectively} \\ 5, \text{ if } k_2,\ 3 \varepsilon \{j_6\} \text{ respectively} \end{matrix} \tag{58}$$

it is concluded that the planes (P_1, P_2, P_4) and (P_1, P_2, P_5) together constitute a determinantal locus for the point-sets $\{j_5\}$ and $\{j_3\}$ respectively. These are the determinantal loci of the satellite-position points observed by two station-quadruples.

(b) <u>Three, four or five station-quadruples carry out all the observations.</u>

It was said above that, if a determinantal locus of the previous case of two station-quadruples occurs, then that is also a determinantal locus for any other observational pattern with the same satellite-position points. But if the previous case is non-singular, (i.e., its condensed jacobian matrix is of rank nine), that does <u>not</u> mean that this is the case for the other patterns with the same set of satellite-position points. Thus one has to search for the determinantal loci of the other observational patterns too.

These cases are treated in a similar way. For the determinantal loci of second degree one may partition the set of all 9x9 submatrices of each of these cases in categories as before, while the linear determinantal loci are taken again from the expressions of the determinants of all 9x9 submatrices as polynomials in the fundamental invariants, (see Appendix F).

Consider the case where three station-quadruples carry out all the observations. Without loss of generality take the quadruples $0_5 \equiv (1,2,3,4)$, $0_4 \equiv (1,2,3,4)$ and $0_3 \equiv (1,2,4,5)$. The jacobian matrix of this case is given by the first three row blocks of Table 4. For the set of admissible constraints

$$x_1, y_1, z_1, y_2, z_2, z_3 = \text{constant},$$

leave out the columns corresponding to these coordinates. For the determinantal loci of this case one has to examine the zeroes of the determinants of all 9x9 submatrices. Each 9x9 submatrix of this case has at least one row from each station-quadruple, and it must have at least three rows from any two out of the three quadruples, in order its determinant not be identically equal to zero. Consider as an example the 9x9 submatirx on the following page.

Let $\{j_5\}$, $\{j_4\}$, $\{j_3\}$ be the sets of the satellite-position points observed by the station-quadruples $0_5 \equiv (1,2,3,4)$, $0_4 \equiv (1,2,3,5)$, $0_4 \equiv (1,2,4,5)$ respectively.

Second degree determinantal loci. Working as in the case of two station-quadruples one finds for the general case, (when each one of the three station-quadruples observes six or more satellite-position points), that the second degree determinantal locus consists of three second degree surfaces L_5, L_4, L_3, one for each of the sets $\{j_5\}$, $\{j_4\}$, $\{j_3\}$ respectively. The locus L_5 for the points $\{j_5\}$, passes through

	x_2	x_3	y_3	x_4	y_4	z_4	x_5	y_5	z_5
$0_5 \equiv (1,2,3,4)$	$-(X_1-x_2)\Delta^1_{25}$	$(X_1-x_3)\Delta^1_{35}$	$(Y_1-y_3)\Delta^1_{35}$	$-(X_1-x_4)\Delta^1_{45}$	$-(Y_1-y_4)\Delta^1_{45}$	$-(Z_1-z_4)\Delta^1_{45}$	0	0	0
	$-(X_2-x_2)\Delta^2_{25}$	$(X_2-x_3)\Delta^2_{35}$	$(Y_2-y_3)\Delta^2_{35}$	$-(X_2-x_4)\Delta^2_{45}$	$-(Y_2-y_4)\Delta^2_{45}$	$-(Z_2-z_4)\Delta^2_{45}$	0	0	0
	$-(X_3-x_2)\Delta^3_{25}$	$(X_3-x_3)\Delta^3_{35}$	$(Y_3-y_3)\Delta^3_{35}$	$-(X_3-x_4)\Delta^3_{45}$	$-(Y_3-y_4)\Delta^3_{45}$	$-(Z_3-z_4)\Delta^3_{45}$	0	0	0
$0_4 \equiv (1,2,3,5)$	$-(X_4-x_2)\Delta^4_{24}$	$(X_4-x_3)\Delta^4_{34}$	$(Y_4-y_3)\Delta^4_{34}$	0	0	0	$-(X_4-x_5)\Delta^4_{54}$	$-(Y_4-y_5)\Delta^4_{54}$	$-(Z_4-z_5)\Delta^4_{54}$
	$-(X_5-x_2)\Delta^5_{24}$	$(X_5-x_3)\Delta^5_{34}$	$(Y_5-y_3)\Delta^5_{34}$	0	0	0	$-(X_5-x_5)\Delta^5_{54}$	$-(Y_5-y_5)\Delta^5_{54}$	$-(Z_5-z_5)\Delta^5_{54}$
	$-(X_6-x_2)\Delta^6_{24}$	$(X_6-x_3)\Delta^6_{34}$	$(Y_6-y_3)\Delta^6_{34}$	0	0	0	$-(X_6-x_5)\Delta^6_{54}$	$-(Y_6-y_5)\Delta^6_{54}$	$-(Z_6-z_5)\Delta^6_{54}$
$0_3 \equiv (1,2,4,5)$	$-(X_7-x_2)\Delta^7_{23}$	0	0	$(X_7-x_4)\Delta^7_{43}$	$(Y_7-y_4)\Delta^7_{43}$	$(Z_7-z_4)\Delta^7_{43}$	$-(X_7-x_5)\Delta^7_{53}$	$-(Y_7-y_5)\Delta^7_{53}$	$-(Z_7-z_5)\Delta^7_{53}$
	$-(X_8-x_2)\Delta^8_{23}$	0	0	$(X_8-x_4)\Delta^8_{43}$	$(Y_8-y_4)\Delta^8_{43}$	$(Z_8-z_4)\Delta^8_{43}$	$-(X_8-x_5)\Delta^8_{53}$	$-(Y_8-y_5)\Delta^8_{53}$	$-(Z_8-z_5)\Delta^8_{53}$
	$-(X_9-x_2)\Delta^9_{23}$	0	0	$(X_9-x_4)\Delta^9_{43}$	$(Y_9-y_4)\Delta^9_{43}$	$(Z_9-z_4)\Delta^9_{43}$	$-(X_9-x_5)\Delta^9_{53}$	$-(Y_9-y_5)\Delta^9_{53}$	$-(Z_9-z_5)\Delta^9_{53}$

Table 7. A 9x9 Submatrix of the Jacobian Matrix of the case when the three station-quadruples $0_5 \equiv (1,2,3,4)$, $0_4 \equiv (1,2,3,5)$ and $0_3 \equiv (1,2,4,5)$ carry out all the Observations.

the points: origin, P_1, P_2, P_3, and P_4, and its equation is taken by equating to zero the determinant of any 9x9 submatrix which has only one row from $O_5 \equiv (1,2,3,4)$, say the corresponding to $Q_{j_5} \equiv (X_{j_5}, Y_{j_5}, Z_{j_5})$, that determinant being considered as a functional with respect to $X_{j_5}, Y_{j_5}, Z_{j_5}$. This is a second degree surface which contains the points $\{j_5\}$. Similarly the locus L_4 for the points $\{j_4\}$, passes through the points: origin, P_1, P_2, P_3, and P_5, and its equation is taken by equating to zero the determinant of any 9x9 submatrix which has only one row from $O_4 \equiv (1,2,3,5)$, say the corresponding to $Q_{j_4} \equiv (X_{j_4}, Y_{j_4}, Z_{j_4})$, that determinant being considered as a functional with respect to $X_{j_4}, Y_{j_4}, Z_{j_4}$. At last the locus L_3 for the points $\{j_3\}$, passes through the points: origin, P_1, P_2, P_4, and P_5, and its equation is taken be equating to zero the determinant of any 9x9 submatrix which has only one row from $O_3 \equiv (1,2,4,5)$, say the corresponding to $Q_{j_3} \equiv (X_{j_3}, Y_{j_3}, Z_{j_3})$, that determinant being considered as a functional with respect to $X_{j_3}, Y_{j_3}, Z_{j_3}$. Arguing as in the case of two station-quadruples, one concludes that each of the loci L_5, L_4, L_3 is not a general second degree surface, but it is such that any five of its points determine it completely. Since in pairs these loci have four points in common, if there exists a point observed by all five stations, those loci have to coincide with each other.

Linear determinantal loci. The expression of the determinant of any 9x9 submatrix of the jacobian matrix of this case, as a polynomial in the fundamental invariants is the following mess.

$$|\Delta(0_5,0_4,0_3)| = \sum_{\rho_1(\rho_2(\rho_3))} (-1)^{\sigma_1+\sigma_2+\sigma_3} \begin{vmatrix} x_2 & 1 \\ X_{n_1} & 1 \end{vmatrix} \begin{vmatrix} x_3 & y_3 & 1 \\ X_{n_2} & Y_{n_2} & 1 \\ X_{n_3} & Y_{n_3} & 1 \end{vmatrix} \mathfrak{D}_4^{n_4,n_5,n_6} \mathfrak{D}_5^{n_7,n_8,n_9} \Delta_{4\kappa_1}^{n_4} \Delta_{4\kappa_2}^{n_5} \Delta_{4\kappa_3}^{n_6}$$

$$\times \Delta_{5\lambda_1}^{n_7} \Delta_{5\lambda_2}^{n_8} \Delta_{5\lambda_3}^{n_9} \Delta_{2\mu}^{n_1} \Delta_{3\nu_1}^{n_2} \Delta_{3\nu_2}^{n_3} \tag{59}$$

where

$$n_1 \varepsilon \{j_5\} \cup \{j_4\} \cup \{j_3\},$$

$$n_2, n_3 \varepsilon \{j_5\} \cup \{j_4\},$$

$$n_4, n_5, n_6 \varepsilon \{j_5\} \cup \{j_3\},$$

$$n_7, n_8, n_9 \varepsilon \{j_4\} \cup \{j_e\},$$

$$\kappa_1,\ \kappa_2,\ \kappa_3 = \begin{cases} 5, & \text{if } n_4, n_5, n_6 \varepsilon \{j_5\} \text{ respectively,} \\ 3, & \text{if } n_4\ n_5, n_6 \varepsilon \{j_3\} \text{ respectively,} \end{cases}$$

$$\lambda_1,\ \lambda_2,\ \lambda_3 = \begin{cases} 4, & \text{if } n_7, n_8, n_9 \varepsilon \{j_4\} \text{ respectively,} \\ 3, & \text{if } n_7, n_8, n_9 \varepsilon \{j_3\} \text{ respectively,} \end{cases}$$

$$\mu = \begin{cases} 5, & \text{if } n_1 \varepsilon \{j_5\}, \\ 4, & \text{if } n_1 \varepsilon \{j_4\}, \\ 3, & \text{if } n_1 \varepsilon \{j_3\}, \end{cases}$$

$$\nu_1, \nu_2 = \begin{cases} 5, & \text{if } n_2, n_3 \varepsilon \{j_5\} \text{ respectively,} \\ 4, & \text{if } n_2, n_3 \varepsilon \{j_4\} \text{ respectively.} \end{cases}$$

and $\mathfrak{D}_4^{n_4,n_5,n_6}, \mathfrak{D}_5^{n_7,n_8,n_9}$ are given by (49).

By equating to zero each factor of the typical term in the expression (59) one obtains the linear determinantal loci.

(i) From

$$\begin{vmatrix} x_2 & 1 \\ X_{n_1} & 1 \end{vmatrix} = 0, \qquad (n_1 \varepsilon \{j_5\} \cup \{j_4\} \cup \{j_3\}) \tag{60}$$

one concludes that the plane through the point $P_2 \equiv (x_2, y_2, Z_2)$ and perpendicular to the x-axis is a determinantal locus for all the points $\{j_5\} \cup \{j_4\} \cup \{j_3\}$.

(ii) From

$$\begin{vmatrix} x_3 & y_3 & 1 \\ X_{n2} & Y_{n2} & 1 \\ X_{n3} & Y_{n3} & 1 \end{vmatrix} = 0 \qquad (n_2, n_3 \varepsilon \{j_5\} \cup \{j_4\} \tag{61}$$

one concludes that each plane of the pencil of planes with axis the projection line of $P_3 \equiv (x_3, y_3, z_3)$ to the (x,y)-plane is a determinantal locus for the point-set $\{j_5\} \cup \{j_4\}$.

(iii) From

$$\mathfrak{D}_4^{n_4, n_5, n_6} = 0, \quad (n_4, n_5, n_6 \varepsilon \{j_5\} \cup \{j_3\}) \tag{62}$$

one concludes that each plane of the sheaf of planes through the point $P_4 \equiv (x_4, y_4, z_4)$ is a determinantal locus for the point-set $\{j_5\} \cup \{j_3\}$.

(iv) From

$$\mathfrak{D}_5^{n_7, n_8, n_9} = 0, \quad (n_7, n_8, n_9 \{j_4\} \cup \{j_3\}) \tag{63}$$

one concludes that each plane of the sheaf of planes through the point $P_5 \equiv (x_5, y_5, z_5)$ is a determinantal locus for the point-set $\{j_4\} \cup \{j_3\}$.

(v) From either one of the relations

$$n_4, n_5, n_6 \varepsilon \{j_5\} \cup \{j_3\} \tag{64}$$

$$\Delta_{4\kappa_1}^{n4} = 0, \quad \Delta_{4\kappa_2}^{n5} = 0, \quad \Delta_{4\kappa_3}^{n6} = 0 \quad \kappa_1, \kappa_2, \kappa_3 = \begin{array}{l} 5, \text{if } n_4, n_5, n_6 \varepsilon \{j_5\} \text{respectively} \\ 3, \text{if } n_4, n_5, n_6 \varepsilon \{j_3\} \text{respectively} \end{array}$$

one concludes that the two planes (P_1,P_2,P_3) and (P_1,P_2,P_5) constitute simultaneously a determinantal locus for the point-sets $\{j_5\}$ and $\{j_3\}$ respectively.

(vi) From either one of the relations

$$\overset{n_7}{\Delta}_{5\lambda_1} = 0,\ \overset{n_8}{\Delta}_{5\lambda_2} = 0,\ \overset{n_9}{\Delta}_{5\lambda_3} = 0 \quad \begin{cases} n_7,n_8,n_9 \varepsilon \{j_4\} \cup \{j_3\} \\ \lambda_1,\lambda_2,\lambda_3 = \begin{cases} 4, & \text{if } n_7,n_8,n_9 \varepsilon \{j_4\} \text{ respectively,} \\ 3, & \text{if } n_7,n_8,n_9 \varepsilon \{j_3\} \text{ respectively} \end{cases} \end{cases} \tag{65}$$

one concludes that the two planes (P_1,P_2,P_3) and (P_1,P_2,P_4) constitute simultaneously a determinantal locus for the point-sets $\{j_4\}$ and $\{j_3\}$ respectively.

(vii) From

$$n_1 \varepsilon \{j_5\} \cup \{j_4\} \cup \{j_3\}$$

$$\overset{n_1}{\Delta}_{2\mu} = 0 \qquad \mu = \begin{cases} 5, & \text{if } n_1 \varepsilon \{j_5\} \\ 4, & \text{if } n_1 \varepsilon \{j_4\} \\ 3, & \text{if } n_1 \varepsilon \{j_3\} \end{cases} \tag{66}$$

one concludes that the planes (P_1,P_3,P_4), (P_1,P_3,P_5) and (P_1,P_4,P_5) are simultaneously a determinantal locus for the point-sets $\{j_5\}$, $\{j_4\}$, $\{j_3\}$ respectively.

(viii) From either one of the relations

$$\overset{n_3}{\Delta}_{3\nu_1} = 0, \quad \overset{n_3}{\Delta}_{3\nu_2} = 0 \quad \begin{cases} n_2,n_3 \varepsilon \{j_5\} \cup \{j_3\} \\ \nu_1,\nu_2 = \begin{cases} 5, & \text{if } n_2,n_3 \varepsilon \{j_5\} \text{ respectively} \\ 4, & \text{if } n_2,n_3 \varepsilon \{j_4\} \text{ respectively} \end{cases} \end{cases} \tag{67}$$

one concludes that the planes (P_1,P_2,P_4) and (P_1,P_2,P_5) constitute simultaneously a determinantal locus for the point-sets $\{j_5\}$ and $\{j_4\}$ respectively.

For the cases when four or five station-quadruples carry out all the observations, one proceeds in a similar way.

2) Determinantal loci for the station-position points

The only loci determined by the station-position points alone, independently from the satellite-position points are:

(a) The straight lines determined by pairs of station-position points,

(b) The planes determined by triples of station-position points, and

(c) The sphere determined by any quadruple of station-position points.

The sphere determined by any quadruple of station-position points is not a determinantal locus independent from the satellite-position points, as it was seen above while treating the determinantal loci for the satellite-position points. One can realize what would happen if the sphere of any four station-position points were a determinantal locus for the fifth one, independently from the satellite-position points! Then the configuration would be near singularity by fact; since the surface of the earth on which the stations lie is approximately spherical.

For the determinantal loci consisting of straight lines, take first the observational pattern when two station-quadruples, say $0_5 \equiv (1,2,3,4)$ and $0_4 \equiv (1,2,3,5)$, carry out all the observations.

Consider the determinants

$$\Delta_{ik}^{j}$$

in the condensed jacobian matrix of Table 5.

From the determinants

Δ_{25}^{j} , Δ_{24}^{j} (for any j)

of the x_2-column block it is concluded that, if each of the point-triples

$\{P_1,P_3,P_4\}$, $\{P_1,P_3,P_5\}$

is collinear, the x_2-column becomes a zero one. Therefore the straight line (P_1,P_3) is a determinantal locus for P_4 and P_5.

Similarly from the rest column blocks of the condensed jacobian matrix of Table 5 one obtains the following. From the x_3,y_3-column block it is concluded that, if each of the point-triples

$\{P_1,P_2,P_4\}$, $\{P_1,P_2,P_5\}$

is collinear the x_3,y_3-column block becomes a zero one. Therefore the straight line (P_1,P_2) is a determinantal locus for P_4 and P_5.

From the x_4,y_4,z_4-column block and x_5,y_5,z_5-column block, it is concluded that, if the point-triple

$\{P_1,P_2,P_3\}$

is collinear both of the last two column blocks become zero ones. Therefore the straight line determined by any two of the three points P_1,P_2,P_3 is a determinantal locus for the third one.

These are the determinantal loci consisting of straight lines for the specific observational pattern when the two station-quadruples $O_5\equiv(1\ 2,3,4)$ and $O_4\equiv(1,2,3,5)$ carry out all the observations, and under the set of admissible constraints x_1,y_1,z_1,y_2,z_2,z_3 = constant. Note that, if at least one of the coordinates x_1,y_1,z_1 of P_1 were unconstrained, one would have also the straight line (P_2,P_3) as a determinantal locus of P_4 and P_5. Thus the selection of the set of admissible constraints does affect this type of determinantal lock. In the following a <u>general procedure</u> of determining such a determinantal locus is described. This procedure is dictated by the form of the condensed jacobian matrix and it is applicable to any number of stations and any observational pattern. It is observed that each determinantal locus of this kind is associated with a station. In other words there are as many such loci as stations. Consider any station of the network, say P_i. The determinantal locus associated with this station is determined as follows. Let $0_1, 0_2, \ldots, 0_k$ be the station-quadruples where P_i participates in. By excluding P_i, there remains from each one of the station-quadruples $0_1, \ldots, 0_k$ a point-triple. Let these triples be denoted by

$$P_1^1, P_2^1, P_3^1 \ , \quad P_1^2, P_2^2, P_3^2 \ , \ \ldots, \quad P_1^k, P_2^k, P_3^k \ .$$

Let also

$$(\varepsilon_1), \ (\varepsilon_2), \ \ldots, \ (\varepsilon_k)$$

be the straight lines determined by any two of the three points of the above point-triples respectively. Without loss of generality one may take

$$(\varepsilon_1) \equiv (P_1^1, P_2^1), \ (\varepsilon_2) \equiv (P_1^2, P_2^2), \ \ldots, \ (\varepsilon_k) \equiv (P_1^k, P_2^k)$$

Then the straight lines (ε_1), (ε_2), ..., (ε_k) constitute simultaneously a determinantal locus for the points $P_3^1, P_3^2, \ldots, P_3^k$ respectively.

For the case of five stations, since any two station-quadruples have three station-position points in common, and consequently any two point-triples from the above have two points in common, one must have:

$$(\varepsilon_1) \equiv (\varepsilon_2) \equiv \ \ldots \ \equiv (\varepsilon_k).$$

For the determinantal loci consisting of planes, one observes that, if any two station-quadruples are coplanar, then their planes coincide, for they have in common three station-position points. Therefore one may proceed as follows.

Suppose that the five station-position points are coplanar. Consider the determinants

$$\Delta^{j}_{ik}, \quad (i,k=1,2,3,4,5)$$

expanded according to the first row, i.e.

$$\Delta^{j}_{ik} = \alpha_{ik}X_j + \beta_{ik}Y_j + \gamma_{ik}Z_j + \delta_{ik}, \tag{68}$$

where α_{ik}, β_{ik}, γ_{ik}, δ_{ik} are functions of the station-coordinates only, (compare with relations (33).

By putting

$$\rho_{ik} = \varepsilon(\alpha^2_{ik} + \beta^2_{ik} + \gamma^2_{ik})^{1/2} \quad (\varepsilon \equiv \text{sign of } \Delta^{j}_{ik}) \tag{69}$$

one has (compare with (34) and (35))

$$\frac{\Delta^{j}_{ik}}{\rho_{ik}} = d, \quad \text{(for all } i,k \text{ and } j) \tag{70}$$

where d is the distance of $Q_j \equiv (X_j, Y_j, Z_j)$ from the plane of the station-position points.

Now consider the general expression for the condensed jacobian matrix, Table 4, along with the set of admissible constraints introduced previously, (i.e., $x_1, y_1, z_1, y_2, z_2, z_3$ = constant), and effect the following rank-equivalent operations.

(i) Divide each row j_5 of $0_5 \equiv (1,2,3,4)$ by the corresponding $\Delta^{j_5}_{45}$.

(ii) Divide each row j_4 of $0_4 \equiv (1,2,3,5)$ by the corresponding $\Delta^{j_4}_{54}$.

(iii) Divide each row j_3 of $0_3 \equiv (1,2,4,5$ by the corresponding $\Delta^{j_3}_{23}$.

(iv) Divide each row j_2 of $0_2 \equiv (1,3,4,5)$ by the corresponding $\Delta^{j_2}_{32}$.

(v) Divide each row j_1 of $0_1 \equiv (2,3,4,5)$ by the corresponding $\Delta_{21}^{j_1}$.

(vi) Subtract from x_2-column, ρ_{25}/ρ_{45} times the x_4-column.

(vii) Subtract from x_5-column, ρ_{24}/ρ_{54} times the x_5-column.

(viii) Add to x_3-column, ρ_{35}/ρ_{45} times the x_4-column.

(ix) Add to y_3-column, ρ_{35}/ρ_{45} times the y_4-column.

(x) Add to x_3-column, ρ_{34}/ρ_{54} times the x_5-column.

(xi) Add to y_3-column, ρ_{34}/ρ_{54} times the y_5-column.

(xii) Subtract the first row in each station-quadruple from the rest of the rows in it.

Then it is obtained:

	x_2	x_3	y_3	x_4	y_4	z_4	x_5	y_5	z_5
$0_5\equiv(1,2,3,4)$	$(x_2-x_4)\frac{\rho_{25}}{\rho_{45}}$	$(x_4-x_3)\frac{\rho_{35}}{\rho_{45}}$	$(y_4-y_3)\frac{\rho_{35}}{\rho_{45}}$	$-X_{j51}-x_4)$	$-(Y_{j51}-y_4)$	$-(Z_{j51}-z_5)$	0	0	0
	0	0	0	$X_{j51}-X_{j52}$	$Y_{j51}-Y_{j52}$	$Z_{j51}-Z_{j52}$	0	0	0
	...	...	...	...	...	...	...	...	...
$0_4\equiv(1,2,3,5)$	$(x_2-x_5)\frac{\rho_{24}}{\rho_{54}}$	$(x_5-x_3)\frac{\rho_{34}}{\rho_{54}}$	$(y_5-y_3)\frac{\rho_{34}}{\rho_{54}}$	0	0	0	$-(X_{j41}-x_5)$	$-(Y_{j41}-y_5)$	$-(Z_{j41}-z_5)$
	0	0	0	0	0	0	$X_{j41}-X_{j42}$	$Y_{j41}-Y_{j42}$	$Z_{j41}-Z_{j42}$
	...	...	...	...	...	...	...	...	...
$0_3\equiv(1,2,4,5)$	a_3	b_3	c_3	$(X_{j31}-x_4)\frac{\rho_{43}}{\rho_{23}}$	$(Y_{j31}-y_4)\frac{\rho_{43}}{\rho_{23}}$	$(Z_{j31}-z_5)\frac{\rho_{43}}{\rho_{23}}$	$(X_{j31}-x_5)\frac{\rho_{53}}{\rho_{23}}$	$(Y_{j31}-y_5)\frac{\rho_{53}}{\rho_{23}}$	$(Z_{j31}-z_5)\frac{\rho_{53}}{\rho_{23}}$
	0	0	0	$(X_{j32}-X_{j31})\frac{\rho_{43}}{\rho_{23}}$	$(Y_{j32}-Y_{j31})\frac{\rho_{43}}{\rho_{23}}$	$(Z_{j32}-Z_{j31})\frac{\rho_{43}}{\rho_{23}}$	$(X_{j32}-X_{j31})\frac{\rho_{53}}{\rho_{23}}$	$(Y_{j32}-Y_{j31})\frac{\rho_{53}}{\rho_{23}}$	$(Z_{j32}-Z_{j31})\frac{\rho_{53}}{\rho_{23}}$
	...	...	...	...	...	...	...	...	...
$0_2\equiv(1,3,4,5)$	a_2	b_2	c_2	$(X_{j21}-x_4)\frac{\rho_{42}}{\rho_{32}}$	$(Y_{j21}-y_4)\frac{\rho_{42}}{\rho_{32}}$	$(Z_{j21}-z_4)\frac{\rho_{42}}{\rho_{32}}$	$(X_{j21}-x_5)\frac{\rho_{52}}{\rho_{32}}$	$(Y_{j21}-y_5)\frac{\rho_{52}}{\rho_{32}}$	$(Z_{j21}-z_5)\frac{\rho_{52}}{\rho_{32}}$
	0	0	0	$(X_{j22}-X_{j21})\frac{\rho_{42}}{\rho_{32}}$	$(Y_{j22}-Y_{j21})\frac{\rho_{42}}{\rho_{32}}$	$(Z_{j22}-Z_{j21})\frac{\rho_{42}}{\rho_{32}}$	$(X_{j22}-X_{j21})\frac{\rho_{52}}{\rho_{32}}$	$(Y_{j22}-Y_{j21})\frac{\rho_{52}}{\rho_{32}}$	$(Z_{j22}-Z_{j21})\frac{\rho_{52}}{\rho_{32}}$
	...	...	...	...	...	...	...	...	...
$0_1\equiv(2,3,4,5)$	a_1	b_1	c_1	$(X_{j11}-x_4)\frac{\rho_{41}}{\rho_{21}}$	$(Y_{j11}-y_4)\frac{\rho_{41}}{\rho_{21}}$	$(Z_{j11}-z_4)\frac{\rho_{41}}{\rho_{21}}$	$(X_{j11}-x_5)\frac{\rho_{51}}{\rho_{21}}$	$(Y_{j11}-y_5)\frac{\rho_{51}}{\rho_{21}}$	$(Z_{j11}-z_5)\frac{\rho_{51}}{\rho_{21}}$
	0	0	0	$(X_{j12}-X_{j11})\frac{\rho_{41}}{\rho_{21}}$	$(Y_{j12}-Y_{j11})\frac{\rho_{41}}{\rho_{21}}$	$(Z_{j12}-Z_{j11})\frac{\rho_{41}}{\rho_{21}}$	$(X_{j12}-X_{j11})\frac{\rho_{51}}{\rho_{21}}$	$(Y_{j12}-Y_{j11})\frac{\rho_{51}}{\rho_{21}}$	$(Z_{j12}-Z_{j11})\frac{\rho_{51}}{\rho_{21}}$
	...	...	...	...	...	...	...	...	...

Table 8. The general expression for the condensed jacobian matrix for five coplanar stations, transformed by rank-equivalent row-operations under the set of admissible constraints $x_1, y_1, z_1, y_2, z_2, z_3$ = constant.

Note. After effecting the first eleven operations, each of the first three columns in each station-quadruple, has the same entries through out all the rows in that quadruple. Thus after the twelfth operation, the first three columns have zeroes everywhere except in the first row in each station-quadruple.

One verifies immediately that any three of the five non-zero rows of the first three columns in Table 8 are linearly dependent. Consider for example the first three non-zero rows, i.e.,

$$\bar{W}_1 \equiv [(x_2-x_4)\frac{\rho_{25}}{\rho_{45}},\ (x_4-x_3)\frac{\rho_{35}}{\rho_{45}},\ (y_4-y_3)\frac{\rho_{35}}{\rho_{45}}] \tag{71$_1$}$$

$$\bar{W}_2 \equiv [(x_2-x_5)\frac{\rho_{24}}{\rho_{54}},\ (x_5-x_3)\frac{\rho_{34}}{\rho_{54}},\ (y_5-y_3)\frac{\rho_{34}}{\rho_{54}}] \tag{71$_2$}$$

$$\bar{W}_3 \equiv [\quad a_3, \qquad b_3, \qquad c_3 \quad] \tag{71$_3$}$$

where

$$\begin{aligned} a_3 &= x_2 + x_4\frac{\rho_{43}\rho_{25}}{\rho_{23}\rho_{45}} + x_5\frac{\rho_{53}\rho_{24}}{\rho_{23}\rho_{54}} \\ b_3 &= -x_4\frac{\rho_{35}\rho_{43}}{\rho_{45}\rho_{23}} - x_5\frac{\rho_{34}\rho_{53}}{\rho_{54}\rho_{23}} \\ c_3 &= -y_4\frac{\rho_{35}\rho_{43}}{\rho_{45}\rho_{23}} - y_5\frac{\rho_{34}\rho_{53}}{\rho_{54}\rho_{23}} \end{aligned} \tag{72}$$

Note. In order to get these expressions for a_3, b_3, c_3 take into account the relation

$$\rho_{ik} = -\rho_{ki} \tag{73}$$

and the identity

$$\rho_{23}\rho_{45}+\rho_{24}\rho_{35}+\rho_{25}\rho_{43} = 0 \tag{74}$$

The three rows $\bar{W}_1$, $\bar{W}_2$, $\bar{W}_3$ are dependent, since for

$$\lambda_1 = -\frac{43}{23} \quad \text{and} \quad \lambda_2 = -\frac{53}{23} \tag{75}$$

one has

$$\lambda_1\bar{W}_1 + \lambda_2\bar{W}_2 = \bar{W}_3 \tag{76}$$

Therefore when the five stations are coplanar the rank of the matrix of Table 8 is less than nine. That is, the plane determined by any three station-position points is a determinantal locus for the rest of the two station-position points.

1.4.4 The Case of Six and More Ground Stations

In the previous two cases, (i.e., in the cases of four and five ground stations) the discussion was more or less exhaustive because the number of basic patterns of observation was small (1 in the case of four stations, and 6 in the case of five stations). This will not be the case for six and more ground stations, for the number of patterns of observation is large (see Section 1.1.4, relation (7)). However in order to find out the determinantal loci for any pattern of observation in the case of six and more stations, one may follow the same routine work that was followed in the previous cases. That happens because, as in the case of five stations, one can form the condensed jacobian matrix of any pattern of observation in the case of six or more stations by putting row-wise one after the other the condensed jacobian matrices of station-quadruples. For example the condensed jacobian matrix corresponding to the observational pattern when all six stations observe simultaneously a set of satellite-position points, is formed by putting row-wise one after the other (regardless order) the condensed jacobian matrices of any three station-quadruples which have three stations in common. The set of the satellite-position points for each one of those three station-quadruples being the very same set observed by all six stations. Before proceeding further, a list of all possible basic observational patterns in the case of six stations, any observational pattern consists of, is given below for reference purposes, (see Section 1.1.4).

List of the Basic Observational Patterns in the Case of Six Stations					
Quadruples (15)	Observed Point-sets	Fivefolds (6)	Observed Point-sets	Sixfolds (1)	Observed Point-sets
$0_{56} \equiv (1,2,3,4)$	$\{j_{56}\}$	$F_6 \equiv (1,2,3,4,5)$	$\{j_6\}$	$E \equiv (1,2,3,4,5,6)$	$\{j\}$
$0_{46} \equiv (1,2,3,5)$	$\{j_{46}\}$	$F_5 \equiv (1,2,3,4,6)$	$\{j_5\}$		
$0_{45} \equiv (1,2,3,6)$	$\{j_{45}\}$	$F_4 \equiv (1,2,3,5,6)$	$\{j_4\}$		
$0_{36} \equiv (1,2,4,5)$	$\{j_{36}\}$	$F_3 \equiv (1,2,4,5,6)$	$\{j_3\}$		
$0_{35} \equiv (1,2,4,6)$	$\{j_{35}\}$	$F_2 \equiv (1,3,4,5,6)$	$\{j_2\}$		
$0_{34} \equiv (1,2,5,6)$	$\{j_{34}\}$	$F_1 \equiv (2,3,4,5,6)$	$\{j_1\}$		
$0_{26} \equiv (1,3,4,5)$	$\{j_{26}\}$				
$0_{25} \equiv (1,3,4,6)$	$\{j_{25}\}$				
$0_{24} \equiv (1,3,5,6)$	$\{j_{24}\}$				
$0_{23} \equiv (1,4,5,6)$	$\{j_{23}\}$				
$0_{16} \equiv (2,3,4,5)$	$\{j_{16}\}$				
$0_{15} \equiv (2,3,4,6)$	$\{j_{15}\}$				
$0_{14} \equiv (2,3,5,6)$	$\{j_{14}\}$				
$0_{13} \equiv (2,4,5,6)$	$\{j_{13}\}$				
$0_{12} \equiv (3,4,5,6)$	$\{j_{12}\}$				

For the solution of the problem of determining the coordinates of the six stations, the observational pattern must consist at least of two station-quadruples comprising all six stations, (say $0_{56} \equiv (1,2,3,4)$ and $0_{34} \equiv (1,2,5,6)$), and at most all possible ones, i.e., fifteen ($C_4^6 = 6!/4!2! = 15$).

Consider any ovservational pattern in the case of six stations, and let $\{Q_j\}$ be the set of all the satellite-position points associated with this pattern. If all six stations were observing simultaneously the set $\{Q_j\}$, one would have the maximum number of observation equations possible with the set $\{Q_j\}$. That is, the set of the observation equations of any observational pattern in the case of six stations is a subset of the set of the observation equations obtained when all six stations observe simultaneously the same set of satellite-position points. For a unique least-squares solution the rank of the condensed jacobian matrix under an admissible set of constraints must be twelve. Therefore, by considering the condensed jacobian matrix as if all stations observed simultaneously the totality of the satellite-position points, no matter what the actual observational pattern is, and checking its rank, if it is less than twelve, then this is the case for the rank of the condensed jacobian matrix of the actual observational pattern as well. However, if it is equal to twelve, the rank of the condensed jacobian matrix of the actual pattern may or may not be equal to twelve. Note that the rank of the condensed jacobian matrix for six stations can not be greater than twelve, for under an admissible set of constraints it has twelve columns. In other words, if a determinantal locus of the observational pattern when all six stations observe simultaneously a certain set of satellite-position points occurs, then that is also a determinantal locus of any other observational pattern with the same set of satellite-position points. As it was said earlier, the condensed jacobian matrix of the observational pattern when all six stations observe simultaneously a certain set $\{Q_j\}$ of satellite-position points is formed by putting row-wise one after the other (regardless of order) the condensed jacobian matrices of three station-quadruples having in common three stations and each observing the same set $\{Q_j\}$.

From the above discussion, the reason why the observational pattern when three station-quadruples carry out all the observations is so important, becomes obvious. The determinantal loci of this observational pattern are searched in the remainder of the present section.

The symbols to be used are completely analogous to those previously used in the cases of four and five stations. Thus the following symbols will be used:

$\Delta^j_{ik\ell}$: For the determinant of fourth order, whose rows from the top to the bottom are the affine coordinates of the satellite-position point $Q_j \equiv (X_j, Y_j, Z_j, 1)$ and three of the six station-position points except P_i, P_k, P_l, in order of increasing index respectively. For example

$$\Delta^j_{123} \equiv \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_4 & y_4 & z_4 & 1 \\ x_5 & y_5 & z_5 & 1 \\ x_6 & y_6 & z_6 & 1 \end{vmatrix} \tag{77}$$

Also

$$D_i^{m_1,m_2,m_3} \equiv \begin{vmatrix} x_i & y_i & z_i & 1 \\ X_{m_1} & Y_{m_1} & Z_{m_1} & 1 \\ X_{m_2} & Y_{m_2} & Z_{m_2} & 1 \\ X_{m_3} & Y_{m_3} & Z_{m_3} & 1 \end{vmatrix} \tag{78}$$

where

$i = 1,2,3,4,5,6$

and

m_1, m_2, m_3 any triple of indices of the satellite-position points observed by statipn P_i.

The condensed jacobian matrix of the case when three station-quadruples, comprising all six stations, carry out all the observations, is given by the following Table 9. This case includes, as it was said above, that when all six stations observe simultaneously all the satellite-position-points. Take the station-quadruples

$$0_{56} \equiv (1,2,3,4),\ 0_{46} \equiv (1,2,3,4,5),\ 0_{45} \equiv (1,2,3,6)$$

which observe the following point-sets respectively

$$\{j_{56}\}, \quad \{j_{46}\}, \quad \{j_{45}\}.$$

If all six stations observe simultaneously all the satellite-position points, then

$$\{j_{56}\} \equiv \{j_{46}\} \equiv \{j_{45}\} \equiv \{j\}.$$

Adopt as in the previous cases the set of admissible constraints

$$x_1, y_1, z_1, y_2, z_2, z_3 = \text{constant}$$

For the determinantal loci the rank of the matrix of Table 9 is less than twelve. Thus one has to examine the zeroes of the determinants of all 12x12 submatrices. It is repeated again here, that these determinants are invariants with respect to the affine transformations and rational integral functions of the coordinates, therefore they have to be expressible as polynomials in the fundamental invariants (see Appendix F). Indeed one obtains by Laplace expansion, as in the previous cases, the following expression for the determinant of a typical 12x12 submatrix under the above adopted set of constraints.

$$|\Delta(O_{56}, O_{46}, O_{45})| = \sum_{\rho_1(\rho_2(\rho_3(\rho_4)))} (-1)^{\sigma_1+\sigma_2+\sigma_3+\sigma_4} \begin{vmatrix} x_2 & 1 \\ X_{m_1} & 1 \end{vmatrix} \begin{vmatrix} x_3 & y_3 & 1 \\ X_{m_2} & Y_{m_2} & 1 \\ X_{m_3} & Y_{m_3} & 1 \end{vmatrix} \mathcal{D}_4^{m_4, m_5, m_6} \mathcal{D}_5^{m_7, m_8, m_9} \mathcal{D}_6^{m_{10}, m_{11}, m_{12}}$$

$$\times \Delta_{2\lambda\mu}^{m_1} \Delta_{3\nu\xi}^{m_2} \Delta_{3\eta\delta}^{m_3} \Delta_{456}^{m_4} \Delta_{456}^{m_5} \Delta_{456}^{m_6} \Delta_{546}^{m_7} \Delta_{546}^{m_8} \Delta_{546}^{m_9} \Delta_{645}^{m_{10}} \Delta_{645}^{m_{11}} \Delta_{645}^{m_{12}} \tag{79}$$

where

$$\begin{aligned} & m_1 \in \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\} \\ & m_2, m_3 \in \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\} \\ & m_4, m_5, m_6 \in \{j_{56}\} \\ & m_7, m_8, m_9 \in \{j_{46}\} \\ & m_{10}, m_{11}, m_{12} \in \{j_{45}\} \\ & (\lambda\mu), (\nu\xi), (\eta\delta) = \begin{cases} (56), \text{ if } m_1, m_2, m_3 \in \{j_{56}\} & \text{respectively,} \\ (46), \text{ if } m_1, m_2, m_3 \in \{j_{46}\} & \text{respectively,} \\ (45), \text{ if } m_1, m_2, m_3 \in \{j_{45}\} & \text{respectively.} \end{cases} \end{aligned} \tag{80}$$

	x_1	y_1	z_1	x_2	y_2	z_2	x_3	y_3	z_3	x_4	y_4	z_4	x_5	y_5	z_5	x_6	y_6	z_6
O_{56} (1,2,3,4)	$(X_{i_1}-x_1)\Delta_{156}^{i_1}$	$(Y_{i_1}-y_1)\Delta_{156}^{i_1}$	$(Z_{i_1}-z_1)\Delta_{156}^{i_1}$	$-(X_{i_1}-x_2)\Delta_{256}^{i_1}$	$-(Y_{i_1}-y_2)\Delta_{256}^{i_1}$	$-(Z_{i_1}-z_2)\Delta_{256}^{i_1}$	$(X_{i_1}-x_3)\Delta_{356}^{i_1}$	$(Y_{i_1}-y_3)\Delta_{356}^{i_1}$	$(Z_{i_1}-z_3)\Delta_{356}^{i_1}$	$-(X_{i_1}-x_4)\Delta_{456}^{i_1}$	$-(Y_{i_1}-y_4)\Delta_{456}^{i_1}$	$-(Z_{i_1}-z_4)\Delta_{456}^{i_1}$	0	0	0	0	0	0
	$(X_{i_2}-x_1)\Delta_{156}^{i_2}$	$(Y_{i_2}-y_1)\Delta_{156}^{i_2}$	$(Z_{i_2}-z_1)\Delta_{156}^{i_2}$	$-(X_{i_2}-x_2)\Delta_{256}^{i_2}$	$-(Y_{i_2}-y_2)\Delta_{256}^{i_2}$	$-(Z_{i_2}-z_2)\Delta_{256}^{i_2}$	$(X_{i_2}-x_3)\Delta_{356}^{i_2}$	$(Y_{i_2}-y_3)\Delta_{356}^{i_2}$	$(Z_{i_2}-z_3)\Delta_{356}^{i_2}$	$-(X_{i_2}-x_4)\Delta_{456}^{i_2}$	$-(Y_{i_2}-y_4)\Delta_{456}^{i_2}$	$-(Z_{i_2}-z_4)\Delta_{456}^{i_2}$	0	0	0	0	0	0
	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
O_{46} (1,2,3,5)	$(X_{k_1}-x_1)\Delta_{146}^{k_1}$	$(Y_{k_1}-y_1)\Delta_{146}^{k_1}$	$(Z_{k_1}-z_1)\Delta_{146}^{k_1}$	$-(X_{k_1}-x_2)\Delta_{246}^{k_1}$	$-(Y_{k_1}-y_2)\Delta_{246}^{k_1}$	$-(Z_{k_1}-z_2)\Delta_{246}^{k_1}$	$(X_{k_1}-x_3)\Delta_{346}^{k_1}$	$(Y_{k_1}-y_3)\Delta_{346}^{k_1}$	$(Z_{k_1}-z_3)\Delta_{346}^{k_1}$	0	0	0	$-(X_{k_1}-x_5)\Delta_{546}^{k_1}$	$-(Y_{k_1}-y_5)\Delta_{546}^{k_1}$	$-(Z_{k_1}-z_5)\Delta_{546}^{k_1}$	0	0	0
	$(X_{k_2}-x_1)\Delta_{146}^{k_2}$	$(Y_{k_2}-y_1)\Delta_{146}^{k_2}$	$(Z_{k_2}-z_1)\Delta_{146}^{k_2}$	$-(X_{k_2}-x_2)\Delta_{246}^{k_2}$	$-(Y_{k_2}-y_2)\Delta_{246}^{k_2}$	$-(Z_{k_2}-z_2)\Delta_{246}^{k_2}$	$(X_{k_2}-x_3)\Delta_{346}^{k_2}$	$(Y_{k_2}-y_3)\Delta_{346}^{k_2}$	$(Z_{k_2}-z_3)\Delta_{346}^{k_2}$	0	0	0	$-(X_{k_2}-x_5)\Delta_{546}^{k_2}$	$-(Y_{k_2}-y_5)\Delta_{546}^{k_2}$	$-(Z_{k_2}-z_5)\Delta_{546}^{k_2}$	0	0	0
	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
O_{45} (1,2,3,6)	$(X_{\ell_1}-x_1)\Delta_{145}^{\ell_1}$	$(Y_{\ell_1}-y_1)\Delta_{145}^{\ell_1}$	$(Z_{\ell_1}-z_1)\Delta_{145}^{\ell_1}$	$-(X_{\ell_1}-x_2)\Delta_{245}^{\ell_1}$	$-(Y_{\ell_1}-y_2)\Delta_{245}^{\ell_1}$	$-(Z_{\ell_1}-z_2)\Delta_{245}^{\ell_1}$	$(X_{\ell_1}-x_3)\Delta_{345}^{\ell_1}$	$(Y_{\ell_1}-y_3)\Delta_{345}^{\ell_1}$	$(Z_{\ell_1}-z_3)\Delta_{345}^{\ell_1}$	0	0	0	0	0	0	$-(X_{\ell_1}-x_6)\Delta_{645}^{\ell_1}$	$-(Y_{\ell_1}-y_6)\Delta_{645}^{\ell_1}$	$-(Z_{\ell_1}-z_6)\Delta_{645}^{\ell_1}$
	$(X_{\ell_2}-x_1)\Delta_{145}^{\ell_2}$	$(Y_{\ell_2}-y_1)\Delta_{145}^{\ell_2}$	$(Z_{\ell_2}-z_1)\Delta_{145}^{\ell_2}$	$-(X_{\ell_2}-x_2)\Delta_{245}^{\ell_2}$	$-(Y_{\ell_2}-y_2)\Delta_{245}^{\ell_2}$	$-(Z_{\ell_2}-z_2)\Delta_{245}^{\ell_2}$	$(X_{\ell_2}-x_3)\Delta_{345}^{\ell_2}$	$(Y_{\ell_2}-y_3)\Delta_{345}^{\ell_2}$	$(Z_{\ell_2}-z_3)\Delta_{345}^{\ell_2}$	0	0	0	0	0	0	$-(X_{\ell_2}-x_6)\Delta_{645}^{\ell_2}$	$-(Y_{\ell_2}-y_6)\Delta_{645}^{\ell_2}$	$-(Z_{\ell_2}-z_6)\Delta_{645}^{\ell_2}$
	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

Table 9. The Condensed Jacobian Matrix for Six Stations when Three Station-Quadruples, O_{56}, O_{46} and O_{45}, Carry out all the Observations.

1) Determinantal Loci for the Satellite-Position Points

Loci of second degree. Each 12x12 submatrix of the jacobian matrix must have at least three rows from each of the three station-quadruples and at most six rows. Working as in the case of five stations, one has in the general case when each of the three station-quadruples observes six and more satellite-position points the following:

The second degree determinantal locus consists of three lock L_{56}, L_{46} and L_{45} for the sets of the satellite-position points $\{j_{56}\}$, $\{j_{46}\}$, and$\{j_{45}\}$ respectively. The locus L_{56} for the point-set $\{j_{56}\}$is a second degree surface passing through the points: origin, P_1,P_2,P_3,P_4 and any two points from $\{j_{56}\}$. In other words L_{56} is not a general second degree surface, but it is such that any seven of its points determine it completely. The equation of L_{56} is taken by equating to zero the determinant of any 12x12 submatrix which has three rows from the station-quadruple $0_{56}\equiv(1,2,3,4)$, the determinant being considered as a functional with respect to the coordinates of one of the satellite-position points observed by $0_{56}\equiv(1,2,3,4)$, (i.e., one from the set $\{j_{56}\}$). The locus L_{46} for the point-set $\{j_{46}\}$ is a second degree surface passing through the points: origin, P_1,P_2,P_3,P_5 and any two points from $\{j_{46}\}$. Thus L_{46} too, is not a general second degree surface, but it is such that any seven of its points determine it completely. The equation of L_{46} is taken by equating to zero the determinant of any 12x12 submatrix which has three rows from the station-quadruple $0_{46}\equiv(1,2,3,5)$, the determinant being considered as a functional with respect to the coordinates of one of the satellite-position points from the point-set$\{j_{45}\}$. The locus L_{45} for the point-set $\{j_{45}\}$ is a second degree surface passing through the points: origin, P_1,P_2,P_3,P_6 and any two points from $\{j_{45}\}$. Therefore L_{45} is not a general second degree surface but it is such that any seven of its points determine it completely. The equation of L_{45} is taken by equating to zero the determinant of any 12x12 submatrix which has three rows from the station-quadruple $O_{45} \equiv (1,2,3,6)$, the determinant being considered as a functional with respect to the coordinates of one of the satellite-position points from the point-set $\{j_{45}\}$.

Note. It is stressed that the loci L_{56}, L_{46}, L_{45} go together. That means that if

the points $\{j_{56}\}$ lie on L_{56},

the points $\{j_{46}\}$ lie on L_{46}, and

the points $\{j_{45}\}$ lie on L_{45}

simultaneously, then the rank of the jacobian matrix is less than twelve. In other words the three loci L_{56}, L_{46}, and L_{45} constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ respectively.

In order to detect the existence of that determinantal locus one may determine each of the loci L_{56}, L_{46}, L_{45} as the surface which passes through nine of its points and verify that the same surface is determined by any seven of its points. Take for example the locus L_{56} for the point-set $\{j_{56}\}$. This is a second degree surface which contains the points: origin, P_1,P_2,P_3,P_4 and $\{j_{56}\}$, in case of determinantal locus. The locus L_{56} in general has the form

$$a_{11}X^2+a_{22}Y^2+a_{33}Z^2+2a_{12}XY+2a_{13}XZ+2a_{23}YZ+2a_{14}X+2a_{24}Y+2a_{34}Z+a_{44} = 0$$

and any nine of its point not in singular configuration are sufficient to determine the coefficients a_{ij}. If the result of the estimation of a_{ij} is such that two of them are identically zero, that means that any seven of the points of L_{56} determine it completely. In case of determinantal locus this must be the case for all three L_{56}, L_{46}, and L_{45}

Linear loci. For the linear loci one has to carry out the same routine work, as in the previous cases, that is to equate to zero each factor of the typical term in the expression (79). Thus one has

(i) From

$$\begin{vmatrix} x_2 & 1 \\ X_{m_1} & 1 \end{vmatrix} = 0, \quad (m_1 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\}) \tag{81}$$

it is concluded that the plane through the point $P_2 \equiv (x_2, y_2, z_2)$ and perpendicular to the x-axis is a determinantal locus for all the satellite-position points, i.e., $\{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\}$.

(ii) From

$$\begin{vmatrix} x_3 & y_3 & 1 \\ X_{m_2} & Y_{m_2} & 1 \\ X_{m_3} & Y_{m_3} & 1 \end{vmatrix} = 0, \quad (m_2, m_3 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\} \tag{82}$$

it is concluded that each plane of the pencil of planes with axis the projection line of $P_3 \equiv (x_3, y_3, z_3)$ to the (x,y)-plane, is a determinantal locus for all the satellite-position points $(\{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\})$:

(iii) From

$$\mathcal{D}_4^{m_4, m_5, m_6} = 0, \quad (m_4, m_5, m_6 \varepsilon \{j_{56}\}) \tag{83}$$

it is concluded that each plane of sheaf of planes through the point $\{x_4, y_4, z_4\}$ is a determinantal locus for the points $\{j_{56}\}$.

(iv) From

$$\mathcal{D}_5^{m_7, m_8, m_9} = 0, \quad (m_7, m_8, m_9 \varepsilon \{j_{46}\}) \tag{84}$$

it is concluded that each plane of the sheaf of planes through the point $P_6 \equiv (x_6, y_6, z_6)$, is a determinantal locus for the points $\{j_{45}\}$.

(v) From

$$\mathfrak{D}_6^{m_{10},m_{11},m_{12}} = 0, \quad (m_{10},m_{11},m_{12}\varepsilon\{j_{45}\}) \tag{85}$$

it is concluded that each plane of the sheaf of planes through the point $P_6 \equiv (x_6,y_6,z_6)$,{is a determinantal locus for the points $\{j_{45}\}$.

(vi) From

$$\Delta_{2\lambda\mu}^{m_1} = 0, \qquad \begin{cases} m_1\varepsilon\ \{j_{56}\}\cup\{j_{46}\}\cup\{j_{45}\} \\ (\lambda\mu)=\begin{cases}(56), \text{ if } m_1\varepsilon\ \{j_{56}\} \\ (46), \text{ if } m_1\varepsilon\ \{j_{46}\} \\ (45), \text{ if } m_1\varepsilon\ \{j_{45}\}\end{cases}\end{cases} \tag{86}$$

it is concluded that the planes (P_1,P_3,P_4), (P_1,P_3,P_5) and (P_1,P_3,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ respectively.

(vii) From either one of the relations

$$m_2,m_3\varepsilon\{j_{56}\}\cup\{j_{46}\}\cup\{j_{45}\} \tag{87}$$

$$\Delta_{3\nu\xi}^{m_3} = 0, \ \Delta_{3\eta\vartheta}^{m_3} = 0, \ (\nu\xi),(\eta\vartheta) = \begin{cases}(56), \text{ if } m_2,m_3\ \varepsilon\ \{j_{56}\} \text{ respectively} \\ (46), \text{ if } m_2,m_3\ \varepsilon\ \{j_{46}\} \text{ respectively} \\ (45), \text{ if } m_2,m_3\ \varepsilon\ \{j_{45}\} \text{ respectively}\end{cases}$$

it is concluded that the planes (P_1,P_2,P_4), (P_1,P_2,P_5) and (P_1,P_2,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ respectively.

(viii) From either one of the relations

$$\Delta_{456}^{m_4} = 0, \ \Delta_{456}^{m_5} = 0, \ \Delta_{456}^{m_6} = 0, \ (m_4,m_5,m_6\ \varepsilon\ \{j_{56}\}) \tag{88}$$

it is concluded that the plane (P_1,P_2,P_3) is a determinantal locus for the points $\{j_{56}\}$.

(ix) From either one of the relations

$$\Delta_{546}^{m_7} = 0, \ \Delta_{546}^{m_8} = 0, \ \Delta_{546}^{m_9} = 0, \ (m_7,m_8,m_9\ \varepsilon\ \{j_{46}\}) \tag{89}$$

it is concluded that the plane (P_1, P_2, P_3) is a determinantal locus for the points $\{j_{46}\}$.

(x) From either one of the relations

$$\Delta_{645}^{m_{10}} = 0, \; \Delta_{645}^{m_{11}} = 0, \; \Delta_{645}^{m_{12}} = 0, \; (m_{10}, m_{11}, m_{12} \varepsilon \{j_{45}\} \qquad (90)$$

it is concluded that the plane (P_1, P_2, P_3) is a determinantal locus for the points $\{j_{45}\}$.

2) Determinantal Loci for the Station-Position Points

As in the previous cases of four and five stations, one is interested in finding out determinantal loci for the station-position points independently from the satellite-position points. From the discussion of the determinantal loci for satellite-position points, it is concluded that there is no second degree determinantal locus for the station-position points independent from the satellite-position points. Therefore one proceeds to examine the linear ones, i.e., those consisting of straight lines or planes.

The determinantal loci consisting of straight lines, are obtained in a way similar to that which was followed in the case of five stations. The general procedure which was described there, applies here although it is not needed, for the condensed jacobian matrix of the observational pattern under discussion is available. Thus, under the set of admissible constraints $x_1, y_1, z_1, y_2, z_2, z_3$ = constant, one has the following.

(i) The straight line determined by P_1 and P_3 is a determinantal locus for P_4, P_5 and P_6.

(ii) The straight line determined by P_1 and P_2 is a determinantal locus for P_4, P_5 and P_6.

(iii) The straight line determined by any two of the three points P_1, P_2, P_3 is a determinantal locus for the third one.

If at least one of the coordinates x_1, y_1, z_1 of P_1 were unconstrained, then the straight line (P_2, P_3) would be also a determinantal locus for P_4, P_5 and P_6.

Note: A general procedure of determining such a determinantal locus for any number of stations and any observational pattern, was given within the discussion of determinantal loci for station-position points in the case of five stations.

For the determinantal loci consisting of planes one observes that, since in this observational pattern the three station-quadruples involved, $(0_{56} \equiv (1,2,3,4)$, $0_{46} \equiv (1,2,3,5)$, $0_{45} \equiv (1,2,3,6)$ have in common the three station-position points P_1, P_2 and P_3, if each of these quadruples is coplanar, then their planes coincide. Therefore one may start off searching for these loci by assuming all six station-position points are coplanar.

Assuming that the six station-position points are coplanar, consider the determinants

$$\Delta^j_{ik\ell}\ ,\ (i,k,\ell = 1,2,3,4,5,6)$$

expanded according to the first row, i.e.,

$$\Delta_{ik\ell} = \alpha_{ik\ell} X_j + \beta_{ik\ell} Y_j + \gamma_{ik\ell} Z_j + \delta_{ik\ell} \tag{91}$$

where $\alpha_{ik\ell}$, $\beta_{ik\ell}$, $\gamma_{ik\ell}$, $\delta_{ik\ell}$ are functions of the station coordinates only, (compare with (33) and (68)).

By putting

$$\rho_{ik\ell} = \varepsilon(\alpha^2_{ik\ell} + \beta^2_{ik\ell} + \gamma^2_{ik\ell})^{1/2},\ (\varepsilon = \text{sign of } \Delta^j_{ik\ell}) \tag{92}$$

one has

$$\frac{\Delta_{ik\ell}}{\rho_{ik\ell}} = d,\ (\text{for all } i,k,\ell \text{ and } j) \tag{92}$$

where d is the distance of $Q_j \equiv (X_j, Y_j, Z_j)$ from the plane of the station-position points.

Consider now the jacobian matrix of Table 9 along with the set of admissible constraints adopted above, (i.e., $x_1, y_1, z_1, y_2, z_2, z_3$ =constant), and effect the following rank-equivalent operations.

(i) Divide each row j_{56} of $0_{56}\equiv(1,2,3,4)$ by the corresponding $\Delta_{456}^{j_{56}}$

(ii) Divide each row j_{46} of $0_{46}\equiv(1,2,3,5)$ by the corresponding $\Delta_{546}^{j_{46}}$

(iii) Divide each row j_{45} of $0_{45}\equiv(1,2,3,6)$ by the corresponding $\Delta_{645}^{j_{45}}$

(iv) Subtract from x_2-column, ρ_{256}/ρ_{456} times the x_4-column

(v) Subtract from x_2-column, ρ_{246}/ρ_{546} times the x_5-column

(vi) Subtract from x_2-column, ρ_{246}/ρ_{645} times the x_6-column

(vii) Add to x_3-column, ρ_{356}/ρ_{456} times the x_4-column

(viii) Add to y_3-column, ρ_{356}/ρ_{456} times the y_4-column

(ix) Add to x_3-column, ρ_{346}/ρ_{546} times the x_5-column

(x) Add to y_3-column, ρ_{346}/ρ_{546} times the y_5-column

(xi) Add to x_3-column, ρ_{345}/ρ_{645} times the x_6-column

(xii) Add to y_3-column, ρ_{345}/ρ_{645} times the y_6-column

(xiii) Subtract the first row in each station-quadruple from the rest rows in it.

Then it is obtained the following rank equivalent matrix, (Table 10).

The column-block of the first three columns of Table 10 is independent of the satellite-position points and has three non-zero rows, namely

$$\bar{W}_1\equiv[(x_2-x_4)\frac{\rho_{256}}{\rho_{456}},\ (x_4-x_3)\frac{\rho_{356}}{\rho_{456}},\ (y_4-y_3)\frac{\rho_{356}}{\rho_{456}}], \tag{94$_1$}$$

$$\bar{W}_2\equiv[(x_2-x_5)\frac{\rho_{246}}{\rho_{546}},\ (x_5-x_3)\frac{\rho_{346}}{\rho_{546}},\ (y_5-y_3)\frac{\rho_{346}}{\rho_{546}}], \tag{94$_2$}$$

$$\bar{W}_3\equiv[(x_2-x_6)\frac{\rho_{245}}{\rho_{645}},\ (x_6-x_3)\frac{\rho_{345}}{\rho_{645}},\ (y_6-y_3)\frac{\rho_{345}}{\rho_{645}}]. \tag{94$_3$}$$

	x_2	x_3	y_3	x_4	y_4	z_4	x_5	y_5	z_5	x_6	y_6	z_6
$O_{56}\equiv(1,2,3,4)$	$(x_2-x_4)\frac{\rho_{256}}{\rho_{456}}$	$(x_4-x_3)\frac{\rho_{356}}{\rho_{456}}$	$(y_4-y_3)\frac{\rho_{356}}{\rho_{456}}$	$-(X_{i_1}-x_4)$	$-(Y_{i_1}-y_4)$	$-(Z_{i_1}-z_4)$	0	0	0	0	0	0
	0	0	0	$X_{i_1}-X_{i_2}$	$Y_{i_1}-Y_{i_2}$	$Z_{i_1}-Z_{i_2}$	0	0	0	0	0	0
	...	...	...	...	...	...	...	...	...	...	...	...
$O_{46}\equiv(1,2,3,5)$	$(x_2-x_5)\frac{\rho_{246}}{\rho_{546}}$	$(x_5-x_3)\frac{\rho_{346}}{\rho_{546}}$	$(y_5-y_3)\frac{\rho_{346}}{\rho_{546}}$	0	0	0	$-(X_{k_1}-x_5)$	$-(Y_{k_1}-y_5)$	$-(Z_{k_1}-z_5)$	0	0	0
	0	0	0	0	0	0	$X_{k_1}-X_{k_2}$	$Y_{k_1}-Y_{k_2}$	$Z_{k_1}-Z_{k_2}$	0	0	0
	...	...	...	...	...	...	...	...	...	...	...	...
$O_{45}\equiv(1,2,3,6)$	$(x_2-x_6)\frac{\rho_{245}}{\rho_{645}}$	$(x_6-x_3)\frac{\rho_{345}}{\rho_{645}}$	$(y_6-y_3)\frac{\rho_{345}}{\rho_{645}}$	0	0	0	0	0	0	$-(X_{\ell_1}-x_6)$	$-(Y_{\ell_1}-y_6)$	$-(Z_{\ell_1}-z_6)$
	0	0	0	0	0	0	0	0	0	$X_{\ell_1}-X_{\ell_2}$	$Y_{\ell_1}-Y_{\ell_2}$	$Z_{\ell_1}-Z_{\ell_2}$
	...	...	...	...	...	...	...	...	...	...	...	...

Table 10. The Transformed Condensed Jacobian Matrix for Six Coplanar Stations, when the Three Station-Quadruples O_{56}, O_{46}, O_{45} Carry out all the Observations.

The determinant of the matrix of these rows, i.e.,

$$\det\begin{bmatrix}\bar{W}_1\\ \bar{W}_2\\ \bar{W}_3\end{bmatrix} = \frac{1}{\rho_{456}\rho_{546}\rho_{645}}\left[\rho_{256}\rho_{346}\rho_{345}\,(x_x-x_4)\begin{vmatrix}x_3 & y_3 & 1\\ x_5 & y_5 & 1\\ x_6 & y_6 & 1\end{vmatrix}\right.$$

$$-\rho_{246}\rho_{356}\rho_{345}\,(x_2-x_5)\begin{vmatrix}x_3 & y_3 & 1\\ x_4 & y_4 & 1\\ x_6 & y_6 & 1\end{vmatrix} \qquad (95)$$

$$\left.+\rho_{245}\rho_{356}\rho_{346}\,(x_2-x_6)\begin{vmatrix}x_3 & y_3 & 1\\ x_4 & y_4 & 1\\ x_5 & y_5 & 1\end{vmatrix}\right]$$

is not identically zero. Therefore the plane determined by any three station-position points is not in general a determinantal locus for the rest three points.

Since $\det[\bar{W}_1,\bar{W}_2,\bar{W}_3]^T$ is invariant under affine transformations (see Appendix C) by equating it to zero one may interpret it geometrically. As a matter of fact by considering that determinant as a functional with respect to the coordinates of one of the station-position points, say x_6,y_6,z_6, and equating it to zero, one has a second degree planar locus for $P_6\equiv(x_6,y_6,z_6)$. That locus passes through the points P_1,P_2,P_3,P_4,P_5 as it is verified immediately. If the plane of the station-position points has some specific positions with respect to the coordinate system, the determinant (95) is zero no matter where the six points lie in it. Thus if the plane of the station-position points is perpendicular to the (x,y)-plane, whence the projections of the six station-position points on the x,y-plane are collinear, then the three determinants on the right hand side of (95) vanish. Therefore the plane determined by any three station-position points is not a determinantal locus for the other points, unless in addition to coplanarity condition the determinant (95) vanishes. Remark. For any observational pattern in the case of six stations, consisting of at least three station-quadruples, the plane determined by any three station-position points, is not a determinantal locus for the other groups of three.

Suppose now that one has the case of n stations. From the procedure of getting the condensed jacobian matrix, it is concluded that the condensed jacobian matrix of the observational pattern when all n stations observe simultaneously a certain set $\{Q_j\}$ of satellite-position points, is formed by putting row wise one after the other (regardless order) the condensed jacobian matrices of (n-3) station-quadruples which have in common three stations, and each one of which observes the very same set $\{Q_j\}$ of satellite-position points. Therefore the condensed jacobian matrix of any observational pattern (for any $n \geq 4$) is formed by putting row wise one after the other, the condensed jacobian matrices of station-quadruples, and consequently for the determinantal loci one may repeat the same routine work that was followed in the previously treated cases.

From the procedure of finding out the determinantal loci of an observational pattern which was established in the previous discussions via forming the condensed jacobian matrix, one may describe general procedures of finding the determinantal loci of an observational pattern, without having to form the condensed jacobian matrix. These general procedures are described in the following section.

1.5 Synopses

Let

$$\{P_i\} \equiv (P_1, P_2, \ldots, P_n), \quad i \varepsilon \{i\} \equiv (1, 2, \ldots, n)$$

be a set of ground stations, observing in some observational pattern a certain set

$$\{Q_j\} \equiv (Q_1, \ldots, Q_m), \quad j \varepsilon \{j\} \equiv (1, 2, \ldots, m)$$

of satellite-position points.

I. Station Quadruples Composing the Observational Pattern

Each satellite-position point Q_j is observed by λ stations, where

$4 \leq \lambda \leq n$.

The observation of Q_j by λ stations is equivalent to the observation of Q_j by $(\lambda-3)$ station-quadruples formed from the λ stations and having in common three stations. Note that there are

$$C_3^\lambda = \frac{\lambda!}{(\lambda-3)!3!} = \frac{\lambda(\lambda-1)(\lambda-2)}{6}$$

selections of $(\lambda-3)$ station-quadruples formed from λ stations and having in common three stations. Therefore the set of station-quadruples composing an observational pattern is not unique as long as there are satellite-position points observed by more than four stations.

For the total number of the station-quadruples composing an observational pattern one has to group the totallity of the satellite-position points in groups, each one of which is observed by the same stations. Thus, suppose that

$$\{Q_j\}_1, \{Q_j\}_2, \ldots, \{Q_j\}_k$$

be sets of satellite-position points observed by

$$\lambda_1, \quad \lambda_2, \ldots, \lambda_k$$

stations respectively. It is understood that

$$\{Q_j\}_1 \quad \{Q_j\}_2 \quad \cdots \quad \{Q_j\}_k = \{Q_j\} = (Q_1, Q_2, \ldots, Q_m).$$

Then the totality of station-quadruples for that observational pattern is

$$K = (\lambda_1 - 3) + (\lambda_2 - 3) + \cdots + (\lambda_k - 3) = \sum_{i=1}^{k} (\lambda_i - 3)$$

or

$$K = \sum_{i=1}^{k} \lambda_i - 3k.$$

For example, if there are only four stations, then all the satellite-position points are observed by the same four stations and in this case $k=1$, that is

$$\{Q_j\}_1 \equiv \{Q_j\} = (Q_1, Q_2, \ldots, Q_m).$$

The formula for K above gives

$$K = 4-3 = 1.$$

II. The Condensed Jacobian Matrix of a Station-Quadruple

This is given in Table 1 (see Section 1.4.2). The same table is recalled here with a change in the indices of the determinants Δ_i^j. Thus instead of the subindex i which indicates the station whose coordinates do not appear in the determinant, one uses the indices of the stations whose coordinates appear in the determinant. Without loss of generality take the station-quadruple P_1, P_2, P_3, P_4. Then the following identities exist (see relations [27]).

$$\Delta_1^j \equiv \Delta_{234}^j, \Delta_2^j \equiv \Delta_{134}^j, \Delta_3^j \equiv \Delta_{124}^j, \Delta_4^j \equiv \Delta_{123}^j,$$

and Table 1 is written

x_1	y_1	z_1	x_2	y_2	z_2	x_3	y_3	z_3	x_4	y_4	z_4
+	+	+	-	-	-	+	+	+	-	-	-
$(X_j-x_2)\Delta^j_{234}$	$(Y_j-y_1)\Delta^j_{234}$	$(Z_j-z_1)\Delta^j_{234}$	$(X_j-x_2)\Delta^j_{134}$	$(Y_j-y_2)\Delta^j_{134}$	$(Z_j-z_2)\Delta^j_{134}$	$(X_j-x_3)\Delta^j_{124}$	$(Y_j-y_3)\Delta^j_{124}$	$(Z_j-z_3)\Delta^j_{124}$	$(X_j-x_4)\Delta^j_{123}$	$(Y_j-y_4)\Delta^j_{123}$	$(Z_j-z_4)\Delta^j_{123}$

<u>Table 1.</u>

Notes. (1) The condensed jacobian matrix of a station-quadruple has four column-blocks, one for each station, with three columns each block corresponding to the x,y,z-coordinates of the respective station. (2) All three-column blocks have the same form. Thus in general the three-column block corresponding to station P_i in the condensed jacobian matrix of a station-quadruple including P_i, say $P_k, P_\ell, P_i, P_m, (k<\ell<i<m)$ is given by

$$\varepsilon(X_j - x_i)\Delta^j_{k\ell m}, \qquad \varepsilon(Y_j - y_i)\Delta^j_{k\ell m}, \qquad \varepsilon(Z_j - z_i)\Delta^j_{k\ell m},$$

where $\varepsilon = \pm 1$, according to the following stipulation.

$$\varepsilon = \begin{cases} +1, \text{ if } i \text{ is first or third in order among } i,k,\ell,m \\ -1, \text{ if } i \text{ is second or fourth in order among } i,k,\ell,m \end{cases}$$

(3) The condensed jacobian matrix of a station-quadruple has as many rows as satellite-position points observed by that quadruple.

III. The Condensed Jacobian Matrix of any Observational Pattern

The condensed jacobian matrix of any observational pattern is formed by putting row wise one after the other (regardless order) the condensed jacobian matrices of the station-quadruples composing that observational pattern.

A symbolic scheme of the condensed jacobian matrix is given below, which is useful for the description of the general procedures of determining the determinantal loci without having to write down the explicit form of the condensed jacobian matrix. That is established through the following three examples.

Example 1. Let an observational pattern in the case of five stations (P_1, P_2, P_3, P_4) be composed by the station-quadruples

$0_{1234} \equiv (P_1, P_2, P_3, P_4)$, and $0_{1235} \equiv (P_1, P_2, P_3, P_5)$

Then the condensed jacobian matrix may be written in the following symbolic way.

$$\begin{bmatrix} 0_{1234} \\ 0_{1235} \end{bmatrix} \equiv \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & 0 \\ P_1 & P_2 & P_3 & 0 & P_5 \end{bmatrix} .$$

Example 2. Let an observational pattern in the case of six stations $(P_1, P_2, P_3, P_4, P_5, P_6)$ be composed by the station-quadruples

$0_{1234} \equiv (P_1, P_2, P_3, P_4)$, $0_{1256} \equiv (P_1, P_2, P_5, P_6)$, $0_{3456} \equiv (P_3, P_4, P_5, P_6)$.

Then the condensed jacobian matrix may be written in the following symbolic way.

$$\begin{bmatrix} 0_{1234} \\ 0_{1256} \\ 0_{3456} \end{bmatrix} \equiv \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & 0 & 0 \\ P_1 & P_2 & 0 & 0 & P_5 & P_6 \\ 0 & 0 & P_3 & P_4 & P_5 & P_6 \end{bmatrix}$$

Example 3. Let an observational pattern in the case of six stations be composed by the station-quadruples

$0_{1234} \equiv (P_1, P_2, P_3, P_4)$, $0_{1235} \equiv (P_1, P_2, P_3, P_5)$

$0_{1256} \equiv (P_1, P_2, P_5, P_6)$, $0_{2345} \equiv (P_2, P_3, P_4, P_5)$.

Then the symbolic scheme of the condensed jacobian matrix is

$$\begin{bmatrix} 0_{1234} \\ 0_{1235} \\ 0_{1256} \\ 0_{2345} \end{bmatrix} \equiv \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & 0 & 0 \\ P_1 & P_2 & P_3 & 0 & P_5 & 0 \\ P_1 & P_2 & 0 & 0 & P_5 & P_6 \\ 0 & P_2 & P_3 & P_4 & P_5 & 0 \end{bmatrix} .$$

IV. Definition of the Determinantal Loci

Consider an observational pattern involving n stations. Its condensed jacobian matrix has n three-column blocks, i.e., $3n$ columns. The rank of this matrix is at most

$$\mu = 3n-6 = 3(n-2).$$

Therefore for a unique least squares solution of the problem, one has to introduce six relations independent from each other and the rows of the condensed jacobian matrix. In the previous discussion such relations were introduced as follows:

$$x_1, y_1, z_1, y_2, z_2, z_3 = \text{constant}.$$

A determinantal locus is that relation between the coordinates of the satellite and/or station position points for which the rank of the condensed jacobian matrix is less than $\mu = 3n-6 = 3(n-2)$, or that is the same, that which annihilates the determinants of all $(3n-6)x(3n-6)$ submatrices.

Remark: For the determinantal loci one has to search for the zeroes of the determinants of all $(3n-6)x(3n-6)$ submatrices. Without loss of generality adopt the admissible set of constraints.

$$x_1, y_1, z_1, y_2, z_2, z_3 = \text{constant}.$$

Then in order the determinant of a $(3n-6)x(3n-6)$ submatrix not to be identically zero, it must have from each station-quadruple a minimum number of rows. That minimum number of rows, depends upon how many stations out of the four in that quadruple do not participate in any other station-quadruple of the same observational pattern, and how many coordinates of those stations are unconstrained. It varies from six, when two unconstrained stations or three stations whose three appropriate coordinates are constrained, or four stations whose six appropriate coordinates are constrained, do not participate in any other station-quadruple of the same observational pattern, to one when all four stations of the station-quadruple in question participate in at least one more station quadruple of the same observational pattern. This remark is of importance for the second degree determinantal loci for the satellite-position points, (see the following synopsis V).

V. Description of the Determinantal Loci

The determinantal loci are classified as follows.

1. Determinantal loci for the satellite-position points

   1a. Determinantal loci of second degree.

   1b. Linear determinantal loci.

2. Determinantal loci for the station-position points.

   2a. Determinantal loci consisting of planes.

   2b. Determinantal loci consisting of straight lines.

Remark: The determinantal loci for the station-position points are independent from the satellite-position points, and so one does not have second degree determinantal loci for the station-position points.*

In the following, each of the above classes of determinantal loci is described in a general way for any number of stations and any observational pattern, without utilizing the explicit expression of the condensed jacobian matrix, but only the symbolic one, (see synopsis III above).

1. Determinantal loci for the satellite-position points.

   1a. Determinantal loci of second degree.

Let an observational pattern involving n stations

$$P_1, P_2, \ldots, P_n, \quad (n \geq 4)$$

a set of m satellite-position points.

$$\{Q_j\} = (Q_1, Q_2, \ldots, Q_j, \ldots, Q_m), \quad (m \geq [3(n-2)/(n-3)]),$$

and being composed by a certain set of station-quadruples,

*The second degree determinantal loci involve both satellite and station-position points.

$$0_1, 0_2, \ldots, 0_p \quad \begin{cases} [n/4]^{*}+\sigma \le p \le n!/(n-4)!4! \\ \qquad \sigma = \begin{cases} 0, & \text{if } n/4 \text{ is an integer} \\ 1, & \text{if } n/4 \text{ is not an integer} \end{cases} \end{cases}$$

Let also

$$\{Q_j\}_1,\ \{Q_j\}_2, \ldots, \{Q_j\}_p$$

be the sets of the satellite-position points observed by the station-quadruples $0_1, \ldots, 0_p$ respectively. It is understood that these sets constitute a decomposition of the set $\{Q_j\}$, i.e.,

$$\{Q_j\}_1 \cup \{Q_j\}_2 \cup \cdots \cup \{Q_j\}_p = \{Q_j\} = (Q_1, Q_2, \ldots, Q_j, \ldots, Q_m).$$

Then one may determine a set of p second degree surfaces (some of which or even all of them may coincide),

$$S_1, S_2, \ldots, S_p$$

one for each station-quadruple, such that they constitute <u>simultaneously</u> a determinantal locus for the sets of the satellite-position points

$$\{Q_j\}_1,\ \{Q_j\}_2,\ \ldots,\ \{Q_j\}_p$$

respectively. That is, if

$$\{Q_j\}_1 \varepsilon S_1,\ \{Q_j\}_2 \varepsilon S_2,\ \ldots,\ \{Q_j\}_p \varepsilon S_p \ ^{\dagger}$$

then the determinants of all (3n-6)x(3n-6) submatrices of the condensed jacobian matrix vanish, and consequently the rank of that matrix is less than 3n-6.

One now proceeds to describe the procedure of determining the second degree surface

$$S_\nu,\ (\nu=1,2,\ldots,p)$$

corresponding to the station-quadruple 0_ν. For some fixed value of ν let

$$0_\nu \equiv 0_{qrst} \equiv (P_q, P_r, P_s, P_t),\ q,r,s,t \varepsilon (1,2,\ldots,n)$$

*The square brackets indicate the integer part of the bracketed number.
†The symbol ε stands for expression "belongs to".

The determination of the surface S_ν, depends upon the minimum number of rows allowed by the observational pattern to be taken from the quadruple $0_\nu \equiv 0_{qrst}$ while forming the (3n-6)x(3n-6) submatrices, in order that their determinants not be identically zero. That minimum number of rows depends upon how many stations out of the four in that station-quadruple, do not participate in any other station-quadruple of the same observational pattern, and how many coordinates, if any, of those stations are unconstrained. It varies from six to one as follows:

It is six, when (a) two stations of the quadruple without constrained coordinates, or (b) when three stations of the quadruple with three appropriate* coordinates constrained, or (c) when all four stations† of the quadruple with six appropriate coordinates constrained, do not participate in any other station-quadruple of the same observational pattern.

It is five, when two stations of the quadruple, one with unconstrained coordinates, and the other with one out of the three coordinates constrained, do not participate in any other station-quadruple of the same observational pattern. It is four, when two stations of the quadruple, one with unconstrained coordinates, and the other with two out of the three coordinates constrained, do not participate in any other station-quadruple of the same observational pattern.

It is three, when just one station of the quadruple, with unconstrained coordinates, does not participate in any other station-quadruple of the same observational pattern.

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*By appropriate one means that these constraints are members of an admissible set of constraints.

†In that case the quadruple is independent from the others if there exists any.

It is two, when just one station of the quadruple with one out of its three coordinates constrained, does not participate in any other station-quadruple of the same observational pattern.

It is one, when one station of the quadruple with two out of its three coordinates constrained, does not participate in any other station-quadruple of the same observational pattern.

In case all four stations participate in other station-quadruples of the same observational pattern, one may form (3n-6)x(3n-6) submatrices which do not include any row from that station-quadruple. That is why the minimum number of rows from that quadruple is one, even in this case.

After the description of determining the minimum number of rows allowed by the observational pattern to be taken from a station-quadruple 0_ν, while forming the (3n-6)x(3n-6) submatrices, in order their determinants not to be identically zero, one proceeds in the description of determining the second degree surface S_ν, as the locus of the satellite position points

$$\{Q_j\}_\nu$$

observed by the station-quadruple

$$0_\nu \equiv 0_{qrst} \equiv (P_q, P_r, P_s, P_t).$$

The surface S_ν passes through the points: origin, P_q, P_r, P_s, P_t and the satellite-position points observed by the station-quadruple $0_\nu \equiv 0_{qrst}$. Let τ be the minimum number of rows from the station-quadruple 0_ν. Then the surface S_ν is such that it is determined by the points: origin, P_q, P_r, P_s, P_t and any $(\tau-1)$ satellite-position points. Since a general second degree surface entails nine of its points to be determined, it is concluded the following.

(i) For $\tau=6$ or 5, S is a general second degree surface,

(ii) For $\tau=4,3,2,1$, S is not a general second degree surface, but it is such that any eight, seven, six, five of its points respectively determine it completely.

Remark: The determination of a second degree surface by nine of its points, is a very well known problem in analytic geometry. One has to determine the coefficients of the analytic expression of a general second surface, i.e.,

$$a_{11}X^2+a_{22}Y^2+a_{33}Z^2+2a_{12}XY+2a_{13}XZ+2a_{23}YZ+2a_{14}X+2a_{24}Y+2a_{34}Z+a_{44} = 0.$$

If the second degree surface is not a general one, then some of the coefficients in the above expression are identically equal to zero.

Thus the description of the procedure of determining the second degree determinantal locus is complete. Note that for a given observational pattern there exists just one second degree determinantal locus, consisting of the second degree surfaces $S_1,S_2,\ldots,S_\nu$.

1b. Linear determinantal loci

(α) Each plane of the sheaf of planes through a station P_i, with none of the coordinates is constrained, is a determinantal locus for the satellite-position points observed by all the station-quadruples where the station P_i participates in. If one out of the three coordinates of P_i is constrained, say Z_i, then instead of the planes of the sheaf through P_i, one has the planes of the pencil whose axis passes through P_i and is perpendicular to the (x,y)-plane.

If two out of the three coordinates of P_i are constrained, for example y_i,z_i, then instead of the sheaf of planes through P_i, one has the plane through P_i and perpendicular to the x-axis.

(β) Another category of linear determinantal loci for satellite-position points consists of sets of planes determined by the station-position points. Each determinantal locus of this category is associated with a station. In other words there are as many determinantal loci of this category as stations. Consider any station of the network, say P_i. The determinantal locus associated with this station is determined as follows.

Let

$$0_1, 0_2, \ldots, 0_k$$

be the station-quadruples where P_i participates in, and

$$\{Q_j\}_1, \{Q_j\}_2, \ldots, \{Q_j\}_k$$

be the sets of the satellite-position points observed by those station-quadruples respectively. Excluding station-position point P_i, the remaining three station-position points in each one of the station-quadruples $0_1, 0_2, \ldots, 0_k$ determine a plane. Let these planes be denoted by π_1, $\pi_2, \ldots, \pi_k$. Then the planes $\pi_1, \pi_2, \ldots, \pi_k$ constitute simultaneously a determinantal locus for the satellite-position point-sets $\{Q_j\}_1, \{Q_j\}_2, \ldots, \{Q_j\}_k$ respectively. That is, if

$$\{Q_j\}_1 \; \varepsilon \; \pi_1$$

$$\{Q_j\}_2 \; \varepsilon \; \pi_2$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$\{Q_j\}_k \; \varepsilon \; \pi_k$$

there is no unique least squares solution.

Remark: In case of a determinantal locus of this category, say that associated with station P_i, the three columns of the condensed jacobian matrix corresponding to the coordinates x_i, y_i, z_i of P_i become zero ones.

These are the determinantal loci for the satellite-position points. One might add to the above quoted determinantal loci, the totality of the straight lines in space. However, these are included in the above quoted case (α), for then there exists always a plane through the line in space and a station-position point.

2. Determinantal loci for the station-position points.

2a. Determinantal loci consisting of planes.

It has been demonstrated in previous discussions that in the case of four stations, the plane determined by any three out of the four station-position points is a determinantal locus for the fourth one. That is, if the four station-position points are coplanar there is no unique least squares solution.

Also in the case of five stations, the plane determined by any three out of the five station-position points is a determinantal locus for the other two. That is, if the five station-position points are coplanar there is no unique least squares solution.

In the case of six stations it has been demonstrated that, if the observational pattern is composed of at least three station-quadruples there exists in general no determinantal locus consisting of planes. In the specific observational pattern composed by three station-quadruples having in common three stations, the plane determined by any three out of the six station-position points is a determinantal locus for the other three station-position points, provided that plane is perpendicular to the (x,y)-plane, under the admissible set of constraints: $x_1, y_1, z_1, y_2, z_2, z_3$ = constant. A remarkable note may be drawn from this specific case. It is observed that the selection of the coordinate system may create a determinantal locus, which otherwise would not exist. Thus if in the case of six coplanar station-position points, one selects a coordinate system with the (x,y)-plane perpendicular to the plane of the station-position points, then the observational pattern composed by three station-quadruples with three stations in common, and under the set of constraints: $x_1, y_1, z_1, y_2, z_2, z_3$ = constant, does not yield a unique least squares solution. In general this observational pattern (involving six coplanar station-position points and being composed by three station-quadruples with three stations in common), does not yield a unique least squares solution, if in addition to the coplanarity condition, the general second degree planar curve, determined by any five out of the six station-position points, passes through the sixth one, (see Section 1.4.4).

It should be noted that even in the case of six stations, one may encounter observational patterns where do exist determinantal loci for the station-position points consisting of planes. These observational patterns are composed by two station-quadruples. Consider for example the observational pattern whose the condensed jacobian matrix has the following symbolic scheme.

$$\begin{bmatrix} 0_{1234} \\ 0_{1256} \end{bmatrix} \equiv \begin{bmatrix} P_1 & P_2 & P_3 & P_4 & 0 & 0 \\ P_1 & P_2 & 0 & 0 & P_5 & P_6 \end{bmatrix} .$$

For this observational pattern the plane determined by any three out of the four station-position points of either station-quadruple (0_{1234} and/or 0_{1256}) is a determinantal locus for the fourth one.

In a general observational pattern, the plane determined by any three out of the four station-position points of a station-quadruple, is a determinantal locus for the fourth one, if two stations of that quadruple with unconstrained coordinates, or three stations with three appropriate coordinates constrained, or four stations with six appropriate coordinates constrained, do not participate in any other station quadruple of the same observational pattern.

In the observational pattern involving n stations ($n \geq 6$) and being composed by ν station-quadruples ($\nu \geq 3$) which have in common three stations, provided the station-position points are coplanar, the second degree planar curve determined by any five station-position points, is a determinantal locus for the other $(n-5)$ ones.

2b. Determinantal loci consisting of straight lines

Each of these determinantal loci is associated with a station. In other words there are as many determinantal loci of this type as stations. The general procedure of finding out such a determinantal locus is analogous to that which was described above for the satellite-position points.

Consider any station of the network, say P_i. The determinantal locus for station-position points associated with this station is determined as follows.

Let

$$0_1, 0_2, \ldots, 0_k$$

be the station-quadruples in which P_i participates. By excluding P_i, there remains from each one of the station-quadruples $0_1, 0_2, \ldots, 0_k$ a point-triple. Let these triples be denoted by

$$\{P_1^1,\ P_2^1,\ P_3^1\},\ \{P_1^2,\ P_2^2,\ P_3^2\},\ \ldots,\ \{P_1^k,\ P_2^k,\ P_3^k\}.$$

Let also

$$(\varepsilon_1),\ (\varepsilon_2),\ \ldots,\ (\varepsilon_k)$$

be the straight lines determined by any two of the three points of the above point-triples respectively. Without loss of generality one may take

$$(\varepsilon_1)\equiv(P_1^1,P_2^1),\ (\varepsilon_2)\equiv(P_1^2,P_2^2),\ \ldots,\ (\varepsilon_k)\equiv(P_1^k,P_2^k).$$

Then the straight lines (ε_1), (ε_2), ..., (ε_k) constitute simultaneously a determinantal locus for the station-position points P_3^1, ..., P_3^k respectively. That is, if each one of the point-triples

$$\{P_1^1,P_2^1,P_3^1\},\ \{P_1^2,P_2^2,P_3^2\},\ \ldots,\ \{P_1^k,P_2^k,P_3^k\}$$

is collinear, the x_i,y_i,z_i-column block of the condensed jacobian matrix becomes a zero one, and there is no unique least squares solution.

C.Y.

2. DETERMINANTAL LOCI FOR RANGE-DIFFERENCE NETWORKS

2.1 Introductory Remarks

The present discussion refers to the "continuously" measured range-differences. This observational mode is that of geoceiver - a new compact and portable device for satellite tracking.

Let Q_j, $(j=0,1,2,\ldots,m)$ be a set of satellite-position points at the instances t_j, $(j=0,1,2,\ldots,m)$, and P_i, $(i=1,2,3,4,\ldots,n)$ be a set of ground stations observing by means of geoceiver.

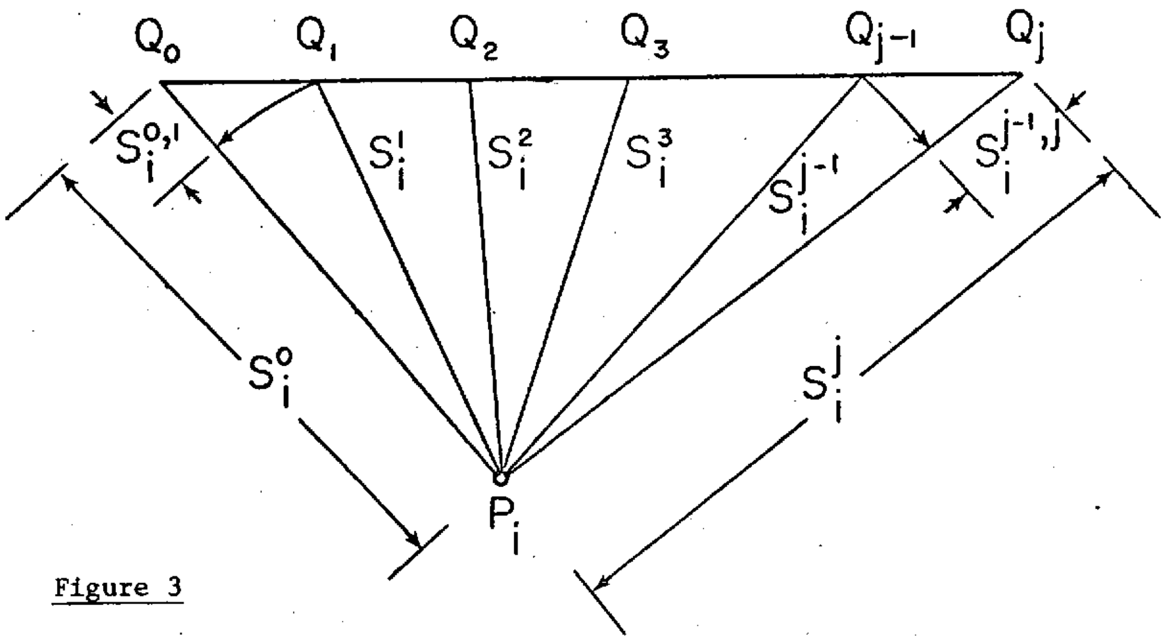

Figure 3

Let also S_i^j be the range from P_i to Q_j, and S_i^{jk} the range difference between the ranges S_i^j and S_i^k, i.e.,

$$S_i^{j,k} = S_i^k - S_i^j \ , \ (k>j) \tag{96}$$

When station P_i observes by means of geoceiver the satellite-position points $Q_0, Q_1, Q_2, \ldots, Q_j, \ldots, Q_m$ (see Figure 3) then the observed quantities are the range differences:

$$S_i^{0,1},\ S_i^{1,2},\ S_i^{2,3},\ \ldots,\ S_i^{j,j-1},\ \ldots,\ S_i^{m,m-1}.$$

It is noted that these symbols stand for the true values of the denoted quantities. Because of the "continuity" one may find the range differences of one range, say $(P_iQ_o) \equiv S_i^0$ from each one of the others. Thus one has

$$S_i^j = S_i^0 + S_i^{0,1} + S_i^{1,2} + \ldots + S_i^{j-1,j} = S_i^0 + \sum_{k=1}^{j} S_i^{k-1,k}\ ,\ (j=1,2,3,\ldots,m) \tag{97}$$

or

$$S_i^{0,j} = S_i^j - S_i^0 = \sum_{k=1}^{j} S_i^{k-1,k}\ ,\ (j=1,2,\ldots,m). \tag{98}$$

The right hand part of (98) is known from the observations. One may substitute the set

$$\{S_i^{0,j}\}_{j=1,2,\ldots m} \equiv \{S_i^{0,1},\ S_i^{1,2},\ S_i^{2,3},\ \ldots,\ S_i^{j-1,j}\ ,\ \ldots,\ S_i^{m-1,m}\}, \tag{99}$$

by the equivalent one

$$\{S_i^{0,j}\}_{j=1,2,\ldots,m} \equiv \{S_i^{0,1},\ S_i^{0,2},\ S_i^{0,3},\ \ldots,\ S_i^{0,j},\ \ldots,\ S_i^{0,m}\}. \tag{100}$$

Indeed the set (99) is equivalent to (100) because if the members of the set (99), (being considered as functional of the coordinates) are algebraiclly independent, this is also the case for the members of the set (100), for each member of (99) is a linear combination of members of (100). As a matter of fact

$$S_i^{j-1,j} = S_i^{0,j} - S_i^{0,j-1} \tag{101}$$

In the following the set $\{S_i^{0,j}\}_{j=1,2,\ldots,m}$ is considered. Let

$$Q_j \equiv (X_j, Y_j, Z_j),\ j=0,1,2,\ldots,m$$

$$P_i \equiv (x_i, y_i, z_i),\ i=1,2,\ldots,n.$$

Then

$$S_i^{0,j} = S_i^j - S_i^0$$
$$= [(X_j-x_i)^2+(Y_j-y_i)^2+(Z_j-z_i)^2]^{1/2} - [(X_0-x_i)^2+(Y_0-y_i)^2+(Z_0-z_i)^2]^{1/2}. \tag{102}$$

Since at each event three unknown coordinates are introduced (i.e., the coordinates of the satellite-position point being observed), at least four stations must observe simultaneously*.

2.2 The Case of Four Ground Stations

Let $P_i \equiv (x_i, y_i, z_i)$, $(i=1,2,3,4)$ be four stations observing simultaneously by means of geoceiver the satellite-position points Q_j, $(j=1,2,\ldots,m)$. Then one has a set of 4m simultaneous equations of the form (102), with 3(m+4) unknowns, six of which are to be constrained in order to establish the coordinate system, so that there actually are 3(m+2) unknowns.

For a unique least squares solution the rank of the jacobian matrix J_M of the 4m functions (102) must be equal to 3(m+2), i.e.,

$$\text{rank } J_M = \mu = 3(m+2). \tag{103}$$

As in the case of range networks, one may condense† the jacobian matrix. The condensed jacobian matrix will be obtained here via elimination (see Appendix E). Take the differential form of (102).

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*This must be the case for geometric satellite geodesy.
†The term condensed jacobian matrix is used conventionally throughout this work, and it is not a general mathematical concept. It is used here on the grounds that the determinants of the submatrices for the determinantal loci (not of any square submatrix) are equal to the corresponding ones of the condensed jacobian matrix.

$$dS_i^{0,j} = \frac{1}{S_i^j}[(X_j-x_i)dX_j+(Y_j-y_i)dY_j+(Z_j-z_i)dZ_j]$$

$$-[(\frac{X_j-x_i}{S_i^j} - \frac{X_0-x_i}{S_i^0})dx_i+(\frac{Y_j-y_i}{S_i^j} - \frac{Y_0-y_i}{S_i^0})dy_i+(\frac{Z_j-z_i}{S_i^j} - \frac{Z_0-z_i}{S_i^0})dz_i],$$

or

$$dS_i^{0,j} = a_{ij}dX_j+b_{ij}dY_j+c_{ij}dZ_j-(a_{ij}-a_{i0})dx_i-(b_{ij}-b_{i0})dy_i-(c_{ij}-c_{i0})dz_i \qquad (104)$$

Note that $dX_0,dY_0,dZ_0 = 0$, for X_0,Y_0,Z_0 may be chosen arbitrarily. However the choice of X_0,Y_0,Z_0 is going to be counted for the constraints in order to establish the coordinate system.

Relations (104) render the mapping of the differential space spanned by $dX_j,dY_j,dZ_j,dx_i,dy_i,dz_i$, $(i=1,2,3,4;j=i,\ldots,m)$ into the differential space spanned by $dS_i^{0,j}$, the matrix of the transformation being the jacobian matrix J_M. Being interested in the rank of the matrix of a linear transformation, it is sufficient to consider the kernel of that transformation.

Thus one takes

$$a_{ij}dX_j+b_{ij}dY_j+c_{ij}dZ_j-(a_{ij}-a_{i0})dx_i-(b_{ij}-b_{i0})dy_i-(c_{ij}-c_{i0})dz_i = 0 \qquad (105)$$

where

a_{ij},b_{ij},c_{ij} - the direction cosines of $\overrightarrow{P_iQ_j}$

a_{i0},b_{i0},c_{i0} - the direction cosines of $\overrightarrow{P_0Q_0}$.

By grouping the 4m equations (105) in m groups, one group per value of the index j, and eliminating from each group the differentials $dX_j,dY_j,$ dZ_j, one comes up with the resultant system of m simultaneous equations, the matrix of the coefficients of which is what has been called condensed jacobian matrix. That is given in Table 12, where

$$\Delta_i^j, \ (i=1,2,3,4;\ j=1,2,..,,m)$$

dx_1	dy_1	dz_1	dx_2	dy_2	dz_2	dx_3	dy_3	dz_3	dx_4	dy_4	dz_4
$(a_{1j}-a_{10})\Delta_1^j$	$(b_{1j}-b_{10})\Delta_1^j$	$(c_{1j}-c_{10})\Delta_1^j$	$-(a_{2j}-a_{20})\Delta_2^j$	$-(b_{2j}-b_{20})\Delta_2^j$	$-(c_{2j}-c_{20})\Delta_2^j$	$(a_{3j}-a_{30})\Delta_3^j$	$(b_{3j}-b_{30})\Delta_3^j$	$(c_{3j}-c_{30})\Delta_3^j$	$-(a_{4j}-a_{40})\Delta_4^j$	$-(b_{4j}-b_{40})\Delta_4^j$	$-(c_{4j}-c_{40})\Delta_4^j$

Table 12. The Condensed Jacobian Matrix for Four Stations "Continuously" Observing Range Differences.

stands for the same determinant as in the case of range networks. To recall it, Δ_3^j is given

$$\Delta_3^j = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} .$$

One notices immediately that the entries of the condensed jacobian matrix are not rational functions of the coordinates of the satellite and station position points. Therefore one can not handle this matrix as that of range networks. To find out the determinantal loci in this case one may proceed as follows. First adopt a set of admissible constraints. Since the constraints X_0, Y_0, Z_0 = constant have already been adopted, three more are needed. By constraining the coordinates y_1, z_1, z_2 one has the following set of admissible constraints.

$$X_0, Y_0, Z_0, y_1, z_1, z_2 = \text{constant.} \tag{106}$$

By leaving out from Table 12 the columns corresponding to dy_1, dz_1, dz_2 and taking the determinant of a typical 9x9 submatrix, it is obtained by Laplace expansion the following expression (compare with (29).

$$|\Delta| = \sum_{\rho_1(\rho_2(\rho_3))} -1^{\rho_1+\sigma_2+\sigma_3} \begin{vmatrix} a_{10} & 1 \\ a_{1j_1} & 1 \end{vmatrix} \begin{vmatrix} a_{20} & b_{20} & 1 \\ a_{2j_2} & b_{2j_2} & 1 \\ a_{2j_3} & b_{2j_3} & 1 \end{vmatrix} \begin{vmatrix} a_{30} & b_{30} & c_{30} & 1 \\ a_{3j_4} & b_{3j_4} & c_{3j_4} & 1 \\ a_{3j_5} & b_{3j_5} & c_{3j_5} & 1 \\ a_{3j_6} & b_{3j_6} & c_{3j_6} & 1 \end{vmatrix} \begin{vmatrix} a_{40} & b_{40} & c_{40} & 1 \\ a_{4j_7} & b_{4j_7} & c_{4j_7} & 1 \\ a_{4j_8} & b_{4j_8} & c_{4j_8} & 1 \\ a_{4j_9} & b_{4j_9} & c_{4j_9} & 1 \end{vmatrix}$$

$$\times\ \Delta_1^{j_1} \Delta_2^{j_2} \Delta_2^{j_3} \Delta_3^{j_4} \Delta_3^{j_5} \Delta_3^{j_6} \Delta_4^{j_7} \Delta_4^{j_8} \Delta_4^{j_9} \tag{107}$$

where

$j_1, j_2, \ldots, j_3$

is any selection of nine satellite-position points out of the m.

The determinantal loci for the satellite and station position points are obtained from the expression (107) by equating to zero each factor of the typical term. Thus one has the following:

(i) From

$$\begin{vmatrix} a_{10} & 1 \\ a_{1j_1} & 1 \end{vmatrix} = 0 \tag{108}$$

it is concluded that if all the satellite-position points, $(Q_0, Q_1, \ldots, Q_m)$, lie on the circular cone with vertex at $P_1 \equiv (x_1, y_1, z_1)$ and axis parallel to the x-axis, (when $a_{10} = a_{1j_1}$, $(j_1 = 1, 2, \ldots, m)$), then the determinants of all 9x9 submatrices vanish. Therefore the circular cone with vertex at $P_1 \equiv (x_1, y_1, z_1)$, axis parallel to the x-axis and passing through the point $Q_0 \equiv (X_0, Y_0, Z_0)$ is a determinantal locus for the satellite-position points $(Q_1, Q_2, \ldots, Q_m)$.

(ii) From

$$\begin{vmatrix} a_{20} & b_{20} & 1 \\ a_{2j_2} & b_{2j_2} & 1 \\ a_{2j_3} & b_{2j_3} & 1 \end{vmatrix} = 0 \tag{109}$$

it is concluded that if all the satellite-position points are distributed on any two semi-straight lines through $P_2 \equiv (x_2, y_2, z_2)$, (which may coincide), then the determinants of all 9x9 submatrices vanish. Therefore any two semi-straight lines through $P_2 \equiv (x_2, y_2, z_2)$, constitute a determinantal locus for the satellite-position points $Q_0, Q_1, \ldots, Q_m$. Since the two semi-straight lines may coincide, any single semi-straight line through P_2, is also a determinantal locus for the satellite-position points $Q_0, Q_1, \ldots, Q_m$

(iii) From

$$\begin{vmatrix} a_{30} & b_{30} & c_{30} & 1 \\ a_{3j_4} & b_{3j_4} & c_{3j_4} & 1 \\ a_{3j_5} & b_{3j_5} & c_{3j_5} & 1 \\ a_{3j_6} & b_{3j_6} & c_{3j_6} & 1 \end{vmatrix} = 0 \tag{110}$$

it is concluded that if all the satellite-position points are distributed on three semi-straight lines through $P_2 \equiv (x_2, y_2, z_2)$, (any two of which or all three may coincide) then the determinants of all 9x9 submatrices vanish. Therefore any three semi-straight lines through $P_3 \equiv (x_3, y_3, z_3)$ constitute a determinantal locus for the satellite-position points $Q_0, Q_1, Q_2, \ldots, Q_m$. Since any two of the three semi-straight lines or all three may coincide, any two or any single semi-straight line through P_3, constitutes also a determinantal locus for the satellite-position points $Q_0, Q_1, Q_2, \ldots, Q_m$.

(iv) From

$$\begin{vmatrix} a_{40} & b_{40} & c_{40} & 1 \\ a_{4j_7} & b_{4j_7} & c_{4j_7} & 1 \\ a_{4j_8} & b_{4j_8} & c_{4j_8} & 1 \\ a_{4j_9} & b_{4j_9} & c_{4j_9} & 1 \end{vmatrix} = 0 \tag{111}$$

it is concluded as above that, any three semi-straight lines through $P_4 \equiv (x_4, y_4, z_4)$, (any two of which or all three may coincide), constitute a determinantal locus for the satellite-position points $Q_0, Q_1, \ldots, Q_m$. Since any two of the three semi-straight lines or all three may coincide, any two or any single semi-straight line through P_4, constitutes also a determinantal locus for the satellite-position points $Q_0, Q_1, \ldots, Q_m$.

(v) From

$$\Delta_1^{j_1} = 0 \tag{112}$$

it is concluded that if all the satellite position points lie in the plane (P_2,P_3,P_4), the condensed jacobian matrix is of rank less than nine and therefore the plane (P_2,P_3,P_4) is a determinantal locus for the set of all the satellite-position points.

(vi) From either one of the relations

$$\Delta_2^{j_2} = 0,\ \Delta_2^{j_3} = 0 \tag{113}$$

it is concluded that the plane (P_1,P_3,P_4) is a determinantal locus for the set of all the satellite-position points.

(vii) From either one of the relations

$$\Delta_3^{j_4} = 0,\ \Delta_3^{j_5} = 0,\ \Delta_3^{j_6} = 0 \tag{114}$$

it is concluded that the plane (P_1,P_2,P_3) is a determinantal locus for the set of all the satellite-position points.

(viii) From either one of the relations

$$\Delta_4^{j_7} = 0,\ \Delta_4^{j_8} = 0,\ \Delta_4^{j_9} = 0 \tag{115}$$

it is concluded that the plane (P_1,P_2,P_3) is a determinantal locus for the set of all the satellite-position points.

For the station-position points one has the following.

By considering the determinants Δ_i^j, i=1,2,3,4 one concludes that, if any three station-position points are collinear, the last three rows in one of these determinants are linearly dependent, and the determinant of any 9x9 submatrix of the condensed jacobian matrix vanishes. In other words the straight line determined by any two station-position points is a determinantal locus for each one of the other two separately. These are the only determinantal loci for the station-position points independently from the satellite-position points. The plane determined by any three station-position points, is not a determinantal locus for the fourth one.

2.3 The Case of More Than Four Ground Stations

When more than four stations participate in the observations, one may consider the condensed jacobian matrix being formed from the condensed jacobian matrices of station-quadruples placed row wise one after the other, in an exactly analogous way as in the case of range networks. Then the determinantal loci for the satellite-position points result by repeating the above quoted for each station quadruple envolved, and the determinantal lock for the station-position points too, result in the same way as in the case of four stations. Thus, the straight line determined by any two of the station-position points, is a determinantal locus for each one of the others separately. In other words if any three station-position points are collinear, the configuration is singular. To derive the determinantal loci of any case formally, one should consider the expression of the determinant of a typical submatrix analogous to (107), and equate each factor of the typical term to zero. Consider for example the case of six stations, $P_i \equiv (x_i, y_i, z_i)$, $(i=1,2,3,4,5,6)$ observing continuously range difference in a pattern of observation such that three station quadruples, say $O_{56} \equiv (1,2,3,4)$, $O_{46} \equiv (1,2,3,5)$ and $Q_{45} \equiv (1,2,3,6)$ carry out all the observations. Let $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ be the satellite-position point sets observed by O56, O46, O45 respectively. Then the determinant of a typical submatrix of the condensed jacobian matrix of this case may be written as follows.

$$|\Delta(0_{56},0_{46},0_{45})| = \sum_{\rho_1(\rho_2(\rho_3(\rho_4(\rho_5))))} (-1)^{\sigma_1+\sigma_2+\sigma_3+\sigma_4} \begin{vmatrix} a_{10} & 1 \\ a_{1m_1} & 1 \end{vmatrix} \begin{vmatrix} a_{20} & b_{20} & 1 \\ a_{2m_2} & b_{2m_2} & 1 \\ a_{2m_3} & b_{2m_3} & 1 \end{vmatrix}$$

$$\times\ D_3^{m_4,m_5,m_6} D_4^{m_7,m_8,m_9} D_5^{m_{10},m_{11},m_{12}} D_6^{m_{13},m_{14},m_{15}}$$

$$\times\ \Delta_{1\alpha\beta}^{m_1} \Delta_{2\gamma\delta}^{m_2} \Delta_{2\varepsilon\zeta}^{m_3} \Delta_{3\eta\Theta}^{m_4} \Delta_{3\kappa\lambda}^{m_5} \Delta_{3\nu\xi}^{m_6}$$

$$\times\ \Delta_{456}^{m_7} \Delta_{456}^{m_8} \Delta_{456}^{m_9} \Delta_{546}^{m_{10}} \Delta_{546}^{m_{11}} \Delta_{546}^{m_{12}} \Delta_{645}^{m_{13}} \Delta_{645}^{m_{14}} \Delta_{645}^{m_{15}} \tag{116}$$

where

$$\begin{aligned} & m_1,m_2,m_3,m_4,m_5,m_6 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\} \\ & m_7,m_8,m_9 \varepsilon \{j_{56}\} \\ & m_{10},m_{11},m_{12} \varepsilon \{j_{46}\} \\ & m_{13},m_{14},m_{15} \varepsilon \{j_{45}\} \end{aligned} \tag{117}$$

$$(\alpha\beta),(\gamma\delta),(\varepsilon\zeta),(\eta\Theta),(\kappa\lambda),(\nu\xi) = \begin{cases} (56), \text{if } m_1,m_2,m_3,m_4,m_5,m_6 \varepsilon \{j_{56}\} \text{ respectively} \\ (46), \text{if } m_1,m_2,m_3,m_4,m_5,m_6 \varepsilon \{j_{46}\} \text{ respectively} \\ (45), \text{if } m_1,m_2,m_3,m_4,m_5,m_6 \varepsilon \{j_{45}\} \text{ respectively} \end{cases}$$

and

$$D_s^{m_i,m_k,m_\ell} = \begin{vmatrix} a_{so} & b_{so} & c_{so} & 1 \\ a_{sm_i} & b_{sm_i} & c_{sm_i} & 1 \\ a_{sm_k} & b_{sm_k} & c_{sm_k} & 1 \\ a_{sm_\ell} & b_{sm_\ell} & c_{sm_\ell} & 1 \end{vmatrix}, \quad (s=3,4,5,6) \tag{118}$$

The determinantal loci are obtained by searching for the zeroes of the typical term of the summation (116).

<u>Determinantal loci for the satellite-position points.</u>

(i) From

$$\begin{vmatrix} a_{10} & 1 \\ a_{1m_1} & 1 \end{vmatrix} = 0, \quad (m_1 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\})$$

it is concluded that the circular cone with vertex at $P_1 \equiv (x_1, y_1, z_1)$, axis parallel to the x-axis and passing through the point $Q_0 \equiv (X_0, Y_0, Z_0)$ is a determinantal locus for the satellite-position points $\{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\}$.

(ii) From

$$\begin{vmatrix} a_{20} & b_{20} & 1 \\ a_{2m_2} & b_{2m_2} & 1 \\ a_{2m_3} & b_{2m_3} & 1 \end{vmatrix} = 0, \quad (m_2, m_3 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\})$$

it is concluded that any two semi-straight lines through $P_2 \equiv (x_2, y_2, z_2)$, (which may coincide) constitute a determinantal locus for the satellite-position points $\{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\}$. Since the two semi-straight lines may coincide, any single semi-straight line through P_2, is also a determinantal locus for the satellite-position points $\{j_{56}\}$ $\{j_{46}\}$ $\{j_{45}\}$.

(iii) From

$$D_3^{m_4, m_5, m_6} = 0, \quad (m_4, m_5, m_6 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\})$$

it is concluded that any three semi-straight lines through $P_3 \equiv (x_3, y_3, z_3)$, (any two of which or all three may coincide), constitute a determinantal locus for the satellite-position points $\{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\}$.

(iv) From

$$D_4^{m_7, m_8, m_9} = 0, \quad (m_7, m_8, m_9 \varepsilon \{j_{56}\})$$

it is concluded as above that, any three semi-straight lines through $P_4 \equiv (x_4, y_4, z_4)$, (any two of which or all three may coincide) constitute a determinantal locus for the satellite-position points $\{j_{56}\}$.

(v) From

$$D_5^{m_{10}, m_{11}, m_{12}} = 0, \quad (m_{10}, m_{11}, m_{12} \varepsilon \{j_{46}\})$$

it is concluded that, any three semi-straight lines through $P_5 \equiv (x_5, y_5, z_5)$, (any two of which or all three may coincide) constitute a determinantal locus for the satellite-position points $\{j_{46}\}$.

(vi) From

$$D_6^{m_{13},m_{14},m_{15}} = 0, \quad (m_{13},m_{14},m_{15} \varepsilon \{j_{45}\})$$

it is concluded that, any three semi-straight lines through $P_6 \equiv (x_6,y_6,z_6)$ (any two of which or all three may coincide) constitute a determinantal locus for the satellite-position points $\{j_{45}\}$.

(vii) From

$$\Delta_{1\alpha\beta}^{m_1} = 0, \quad \begin{cases} m_1 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\} \\ (\alpha\beta) = \begin{cases} (56), \text{ if } m_1 \varepsilon \{j_{56}\} \\ (46), \text{ if } m_1 \varepsilon \{j_{46}\} \\ (45), \text{ if } m_1 \varepsilon \{j_{45}\} \end{cases} \end{cases}$$

it is concluded that, the planes (P_2,P_3,P_4), (P_2,P_3,P_5) and (P_2,P_3,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$ and $\{j_{45}\}$ respectively.

(viii) From either one of the relations

$$\Delta_{2\gamma\delta}^{m_2} = 0, \quad \Delta_{2\varepsilon\zeta}^{m_3} = 0, \quad \begin{cases} m_2, m_3 \varepsilon \{j_{56}\} \cup \{j_{46}\} \cup \{j_{45}\} \\ (\gamma\delta),(\varepsilon\xi) = \begin{cases} (56), \text{ if } m_2,m_3 \varepsilon \{j_{56}\} \\ (46), \text{ if } m_2,m_3 \varepsilon \{j_{46}\} \\ (45), \text{ if } m_2,m_3 \varepsilon \{j_{45}\} \end{cases} \end{cases}$$

it is concluded that the planes (P_1,P_3,P_4), (P_1,P_3,P_5) and (P_1,P_3,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$, and $\{j_{45}\}$ respectively.

(ix) From either one of the relations

$$\Delta^{m_4}_{3\eta 9}=0,\Delta^{m_5}_{3\kappa\lambda}=0,\Delta^{m_6}_{3\nu\xi}=0,\quad \begin{cases} m_4,m_5,m_6\varepsilon\{j_{56}\}\cup\{j_{46}\}\cup\{j_{45}\} \\ (\eta 9),(\kappa\lambda),(\nu\xi)= \begin{cases} (56), \text{ if } m_4,m_5,m_6\varepsilon\{j_{56}\}\text{respectively} \\ (46), \text{ if } m_4,m_5,m_6\varepsilon\{j_{46}\}\text{respectively} \\ (45), \text{ if } m_4,m_5,m_6\varepsilon\{j_{45}\}\text{respectively} \end{cases} \end{cases}$$

it is concluded that the planes (P_1,P_2,P_4), (P_1,P_2,P_5) and (P_1,P_2,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$ and $\{j_{45}\}$ respectively.

(x) From either one of the relations

$$\Delta^{m_7}_{456} = \ , \ \Delta^{m_8}_{456} = 0, \ \Delta^{m_9}_{456} = 0, \ (m_7,m_8,m_9\varepsilon\{j_{56}\})$$

it is concluded that the plane P_1,P_2,P_3 is a determinantal locus for the point-set $\{j_{56}\}$.

(xi) From either one of the relations

$$\Delta^{m_{10}}_{546} = 0, \ \Delta^{m_{11}}_{546} = 0, \ \Delta^{m_{12}}_{546} = 0, \ (m_{10},m_{11},m_{12}\varepsilon\{j_{46}\})$$

it is concluded that the plane (P_1,P_2,P_3) is a determinantal locus for the point-set $\{j_{46}\}$.

(xii) From either one of the relations

$$\Delta^{m_{13}}_{645} = 0, \ \Delta^{m_{14}}_{645} = 0, \ \Delta^{m_{15}}_{645} = 0, \ (m_{13},m_{14},m_{15}\varepsilon\{j_{45}\})$$

it is concluded that the plane (P_1,P_2,P_3) is a determinantal locus for the point set $\{j_{45}\}$. These are the determinantal loci for the satellite-position points.

Determinantal loci for the station-position points

They are obtained by searching for the linear dependence of the last three rows of the determinants $\Delta^{j}_{\lambda\mu\nu}$ in the expression (116). It seems easier to consider these determinants back in their positions in the condensed jacobian matrix. That is not given in this case, for it is of the same form with the corresponding one for range networks (see Table 9), and the following correspondence holds between their entries.

$$
\begin{aligned}
(X_j-x_i)\Delta^j_{\lambda\mu\nu} &\leftrightarrow (a_{ij}-a_{io})\Delta^j_{\lambda\mu\nu} \\
(Y_j-y_i)\Delta^j_{\lambda\mu\nu} &\leftrightarrow (b_{ij}-b_{io})\Delta^j_{\lambda\mu\nu} \qquad (119) \\
(Z_j-z_i)\Delta^j_{\lambda\mu\nu} &\leftrightarrow (c_{ij}-c_{io})\Delta^j_{\lambda\mu\nu}
\end{aligned}
$$

Then one has the following.

(i) By considering the determinants

$$\Delta^j_{156},\ \Delta^j_{146},\ \Delta^j_{145},\ \text{(for any } j),$$

of the first three columns (x_1,y_1,z_1-column block) of the condensed jacobian matrix (refer to Table 9), one concludes that, if each one of the point-triples

$$\{P_2,P_3,P_4\},\ \{P_2,P_3,P_5\},\ \{P_2,P_3,P_6\}$$

is collinear, the first three columns (x_1,y_1,z_1-column block) of the condensed jacobian matrix become zero ones. Therefore the straight line determined by P_2 and P_3 is a determinantal locus for P_4,P_5 and P_6.

Similarly from the rest three-column blocks of the condensed jacobian matrix one obtains the following.

(ii) From the x_2,y_2,z_2-column block it is concluded that, if each one of the point-triples

$$\{P_1,P_3,P_4\},\ \{P_1,P_3,P_5\},\ \{P_1,P_3,P_6\}$$

is collinear, the x_2,y_2,z_2-column block becomes a zero one. Therefore the straight line determined by P_1 and P_3 is a determinantal locus for P_4,P_5 and P_6.

(iii) From the x_3,y_3,z_3-column block, it is concluded that, if each one of the point-triples

$$\{P_1,P_2,P_4\},\ \{P_1,P_2,P_5\},\ \{P_1,P_2,P_6\}$$

is collinear, the x_3,y_3,z_3-column block becomes a zero one. Therefore the straight line determined by P_1 and P_2 is a determinantal locus for P_4,P_5 and P_6.

(iv) From the last three column blocks, i.e.

x_4,y_4,z_4-column block,

x_5,y_5,z_5-column block,

x_6,y_6,z_6-column block

it is concluded that, if the points P_1,P_2,P_3 are collinear, all these three column blocks become zero ones. Therefore the straight line determined by any two of the three points P_1,P_2,P_3 is a determinantal locus for the third one.

Remark: In this observational pattern, it may happen that the point-triples above have two points in common. However this is not the case for all the observational patterns. In order to find out the general form of a determinantal locus of this kind, one should notice that it is associated with one of the stations and is defined through the collinearity conditions of the point-triples which result from the station quadruples in which the associated station participates after excluding that station. The general procedure of finding out such a determinantal locus is given in the following section 2.4. Recapitulating the results of the above observational pattern one has the following.

Determinantal loci for the satellite-position points

(1) Any three semi-straight lines (any two of which or all three may coincide) through either one of the station-position points P_3,P_4,P_5,P_6 (with no coordinates constrained) constitute a determinantal locus for the satellite-position points observed by the station-quadruples,

in which the corresponding station participates.

(2) Any two semi-straight lines (which may coincide) through station P_2 (whose one of the coordinates, z_2, is constrained) constitute a determinantal locus for the satellite-position points observed by the station-quadruples, in which the station P_2 participates.

(3) The circular cone with vertex at the station-position point P_1 (when two of the coordinates, y_1, z_1, are constrained), axis parallel to x-axis and passing through the point $Q_o \equiv (X_o, Y_o, Z_o)$, is a determinantal locus for the satellite-position points observed by the station-quadruples, in which the station P_1 participates.

(4) The plane (P_1,P_2,P_3), is a determinantal locus for each one of the point-sets $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ separately.

(5) The planes (P_2,P_3,P_4), (P_2,P_3,P_5), (P_2,P_3,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ respectively.

(6) The planes (P_1,P_3,P_4), (P_1,P_3,P_5), (P_1,P_3,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ respectively.

(7) The planes (P_1,P_2,P_4), (P_1,P_2,P_5), (P_1,P_2,P_6) constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\}$, $\{j_{46}\}$, $\{j_{45}\}$ respectively.

Determinantal loci for the station-position points

(8) Each one of the straight lines (P_1,P_2), (P_2,P_3) (P_1,P_3) is a determinantal locus for P_4,P_5 and P_6.

(9) The straight line determined by any two of the three points P_1,P_2,P_3 is a determinantal locus for the third one.

Remark: The selected set of admissible constraints $X_o, Y_o, Z_o, y_1, z_1, z_2$ = constant appears to affect the determinantal loci in (2) and (3). Thus, as in the case of range-networks, the selection of the set of admissible constraints does affect the determinantal loci.

2.4 Synopses

After the above discussion one concludes the following about the determinantal loci for networks of ground stations observing range differences by means of geoceiver.

Determinantal loci for satellite-position points

(α) Any three semi-straight lines (any two of which or all three may coincide) through a station-position point $P_i \equiv (x_i, y_i, z_i)$, (with no coordinates constrained) constitute a determinantal locus for the satellite-position points observed by the station-quadruples, in which station P_i participates.

In case one of the coordinates of P_i is constrained, the three semi-straight lines are reduced to two, while in the case when two of the coordinates of P_i are constrained, the determinantal locus is the surface of the circular cone with vertex at P_i, and axis parallel to the coordinate axis corresponding to the unconstrained coordinate.

(β) A second category of determinantal loci for satellite-position points consists of sets of planes determined by the station-position points, (see determinantal loci (4), (5), (6), (7) at the end of the preceding section 2.3). The determination of a determinantal locus of this category, depends upon the observational pattern; however one can give a general procedure of determining a determinantal locus of this category for any observational pattern, without having recourse to the condensed jacobian matrix, (see following Remark). Each determinantal locus of this category is associated with a ground station. In other words there are as many determinantal loci of this category as ground stations. Consider any station of the network, say P_i. The determinantal locus associated with this station is determined as follows:

Let $0_1, 0_2, \ldots, 0_k$ be the station-quadruples in which P_i participates. and $\{Q_j\}_1, \{Q_j\}_2, \ldots, \{Q_j\}_k$ be the sets of satellite-position points observed by these station-quadruples respectively. Excluding station-position point P_i, the remaining three station-position points in each one of the station-quadruples $0_1, 0_2, \ldots, 0_k$ determine a plane. Let these planes be denoted by $\pi_1, \pi_2, \ldots, \pi_k$. Then the planes $\pi_1, \pi_2, \ldots, \pi_k$ constitute simultaneously a determinantal locus for the satellite-position point-sets $\{Q_j\}_1, \{Q_j\}_2, \ldots, \{Q_j\}_k$ respectively. That is, if

$$\{Q_j\}_1 \varepsilon \pi_1$$

$$\{Q_j\}_2 \varepsilon \pi_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\{Q_j\}_k \varepsilon \pi_k$$

there is no unique least squares solution.

One may check the procedure with the observational pattern of the preceding section 2.3. In that observational pattern the three station-quadruples $0_{56}, 0_{46}$ and 0_{45} carry out all the observations, observing the sets of satellite-position points $\{j_{56}\}, \{j_{46}\}$ and $\{j_{45}\}$ respectively. The determinantal locus of the category under discussion, associated with station P_1 for example, according to the above described procedure, is determined as follows. The station P_1 participates in all three station-quadruples $0_{56}, 0_{46}$ and 0_{45}. By excluding P_1, the remaining three station-position points in each one of the station-quadruples O_{56}, O_{46}, O_{45} determine the planes

$$(P_2,P_3,P_4),\ (P_2,P_3,P_5),\ (P_2,P_3,P_6)$$

respectively. These planes constitute simultaneously a determinantal locus for the point-sets $\{j_{56}\},\{j_{46}\},\{j_{45}\}$ respectively. This checks with the result of the preceding section 2.3. As a second check, take the determinantal locus associated with the station P_4. This station participates in the station-quadruple $O_{56} \equiv (1,2,3,4)$ only. By excluding P_4, the remaining three station-position points determine the plane

$$(P_1,P_2,P_3).$$

This plane is a determinantal locus for the satellite-position points $\{j_{56}\}$ observed by the station-quadruple $O_{56} \equiv (1,2,3,4)$. This result checks also with that of the preceding section 1.3.

Remark: In case of a determinantal locus of this category, say that associated with station P_i, the three columns corresponding to the coordinates x_i, y_i, z_i of P_i, of the condensed jacobian matrix become zero ones.

Determinantal loci for station-position points

Each of these determinantal loci is associated with a station. In other words there are as many determinantal loci of this type as stations. The general procedure of finding out such a determinantal locus is analogous to that which was described above for the satellite-position points. Consider any station of the network, say P_i. The determinantal locus for station-position points associated with this station is determined as follows.

Let $O_1, O_2, \ldots, O_k$ be the station-quadruples in which P_i participates. By excluding station-position point P_i, there remains from each one of the station-quadruples $0_1, 0_2, \ldots, 0_k$ a point-triple. Let these triples be denoted* by

$$P_1^1, P_2^1, P_3^1 \ , \quad P_1^2, P_2^2, P_3^2 \ , \ \ldots, \quad P_1^k, P_2^k, P_3^k \ .$$

Let also

$$(\varepsilon_1), \ (\varepsilon_2), \ \ldots, \ (\varepsilon_k)$$

be the straight lines determined by any two of the three points of the above point-triples respectively. Without loss of generality one may take

$$(\varepsilon_1) \equiv (P_1^1, P_2^1), \ (\varepsilon_2) \equiv (P_1^2, P_2^2), \ \ldots, \ (\varepsilon_k) \equiv (P_1^k, P_2^k)$$

Then the straight lines $(\varepsilon_1), (\varepsilon_2), \ldots, (\varepsilon_k)$ constitute simultaneously a determinantal locus for the station-position points $P_3^1, P_3^2, \ldots, P_3^k$ respectively. That is, if each one of the point-triples

$$\{P_1^1, P_2^1, P_3^1\}, \ \{P_1^2, P_2^2, P_3^2\}, \ \ldots, \ \{P_1^k, P_2^k, P_3^k\}$$

is collinear, the x_i, y_i, z_i-column block† of the condensed jacobian matrix becomes a zero one, and there is no unique least squares solution.

Remark: It is concluded from relation (97) that, if one knows the n ranges $S_i^\circ (i=1,2,\ldots,n;$ n=the number of the stations), then the problem is reduced to one of ranges.

*For a specific observational pattern these triples are completely determined. The above notation is used in order to facilitate the general discussion for any observational pattern.

†x_i, y_i, z_i are the coordinates of the station-position point P_i.

APPENDICES

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APPENDIX A

Implicit Function Theorem

The implicit function theorem is stated here without proof. It is given in the same form as in [Carathéodary, 1965], (p.10).

Let n functions $f_1, f_2, \ldots, f_n$ be given depending on the variables $X_j (j=1,2,\ldots,n)$ and $t_\alpha (\alpha=1,2,\ldots,m)$ and being continuous together with all their first partial derivatives

$$\frac{\partial f_i}{\partial x_j}, \frac{\partial f_i}{\partial t_\alpha} \tag{A1}$$

in the (n+m) variables (X_j, t_α).

At an interior point $(X_j^\circ, t_\alpha^\circ)$ in the domain of definition of f_i the equations

$$f_i(X_j^\circ, t_\alpha^\circ) = 0, \; i=1,2,\ldots,n \tag{A2}$$

are all satisfied and also, the determinant

$$D = \left|\frac{\partial f_i^\circ}{\partial X}\right| \neq 0 \tag{A3}$$

There then exists in a certain neighborhood of the point (t_α°) one and only one system of continuous functions

$$X_i = \phi_i(t_\alpha), \; (i=1,2,\ldots,n), \tag{A4}$$

which, first, satisfies the conditions

$$X_i^\circ = \phi_i(t_\alpha^\circ), \; (i=1,2,\ldots,n) \tag{A5}$$

and second, satisfies identically the relations

$$f_i(\phi_i(t_\alpha), t_\beta) = 0, \; (i,j=1,2,\ldots n; \; , \; =1,2,\ldots,m). \tag{A6}$$

The first derivatives $\partial\phi_i/\partial t_\alpha$ exist in the same region, are continuous functions of t_α and are found by solving the system of equations

$$\frac{\partial fi}{\partial Xj}\frac{\partial \phi j}{\partial t\alpha}+\frac{\partial fi}{\partial t\alpha}=0,\ (i=1,2,\ldots,n). \tag{A7}$$

As an example consider the functions of the left-hand sides of (1), for i=1,2,3.

$$\begin{aligned}
f_1(X_j,Y_j,Z_j,x_1,y_1,z_1) &= [(X_j-x_1)^2+(Y_j-y_1)^2+(Z_j-z_1)^2] \\
f_2(X_j,Y_j,Z_j,x_2,y_2,z_2) &= [(X_j-x_2)^2+(Y_j-y_2)^2+(Z_j-z_2)^2] \\
f_3(X_j,Y_j,Z_j,x_3,y_3,z_3) &= [(X_j-x_3)^2+(Y_j-y_3)^2+(Z_j-z_3)^2]
\end{aligned} \tag{A8}$$

The correspondence in notation between this example and the statement of the theorem above is as follows.

$$X_j,\ Y_j,\ Z_j,\ x_1,\ y_1,\ z_1,\ x_2,\ y_2,\ z_2,\ x_3,\ y_3,\ z_3$$

$$X_1,\ X_2,\ X_3,\ t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7,\ t_8,\ t_9.$$

The above functions are continuous with continuous first derivatives; in fact they are analytic. Let $P\equiv(X_j^\circ,Y_j^\circ,Z_j^\circ,x_1^\circ,y_1^\circ,z_1^\circ,x_2^\circ,y_2^\circ,z_2^\circ,x_3^\circ,y_3^\circ,z_3^\circ)$ be a point which annuls f_1,f_2, and f_3, i.e.,

$$\begin{aligned}
f_1(X_j^\circ,Y_j^\circ,Z_j^\circ,x_1^\circ,y_1^\circ,z_1^\circ) &= 0 \\
f_2(X_j^\circ,Y_j^\circ,Z_j^\circ,x_2^\circ,y_2^\circ,z_2^\circ) &= 0 \\
f_3(X_j^\circ,Y_j^\circ,Z_j^\circ,x_3^\circ,y_3^\circ,z_3^\circ) &= 0
\end{aligned} \tag{A9}$$

and suppose that

$$D=\left|\frac{\partial fi}{\partial X_j}\right|_P = 8\begin{vmatrix} X_j^\circ-x_1^\circ & Y_j^\circ-y_1^\circ & Z_j^\circ-z_1^\circ \\ X_j^\circ-x_2^\circ & y_j^\circ-y_2^\circ & Z_j^\circ-z_2^\circ \\ X_j^\circ-x_3^\circ & Y_j^\circ-y_3^\circ & Z_j^\circ-z_3^\circ \end{vmatrix} \neq 0 \tag{A10}$$

Then according to the above theorem in a certain neighborhood of the point $(x_1^\circ,y_1^\circ,z_1^\circ,x_2^\circ,y_2^\circ,z_2^\circ,x_3^\circ,y_3^\circ,z_3^\circ)$ there exists one and only one system of continuous functions:

$$
\begin{aligned}
X_j &= \phi_1(x_1,y_1,z_1,x_2,y_2,z_2,x_3,y_3,z_3) \\
Y_j &= \phi_2(x_1, \ _1,z_1,x_2,y_2,z_2,x_3,y_3,z_3) \\
Z_j &= \phi_3(x_1,y_1,z_1,x_2,y_2,z_2,x_3,y_3,z_3)
\end{aligned}
\tag{A11}
$$

which satisfy identically the relations

$$
\begin{aligned}
f_1(\phi_1,\phi_2,\phi_3,x_1,y_1,z_1) &= 0 \\
f_2(\phi_1,\phi_2,\phi_3,x_2,y_2,z_2) &= 0 \\
f_3(\phi_1,\phi_2,\phi_3,x_3,y_3,z_3) &= 0
\end{aligned}
\tag{A12}
$$

In other words one may solve the system (1).

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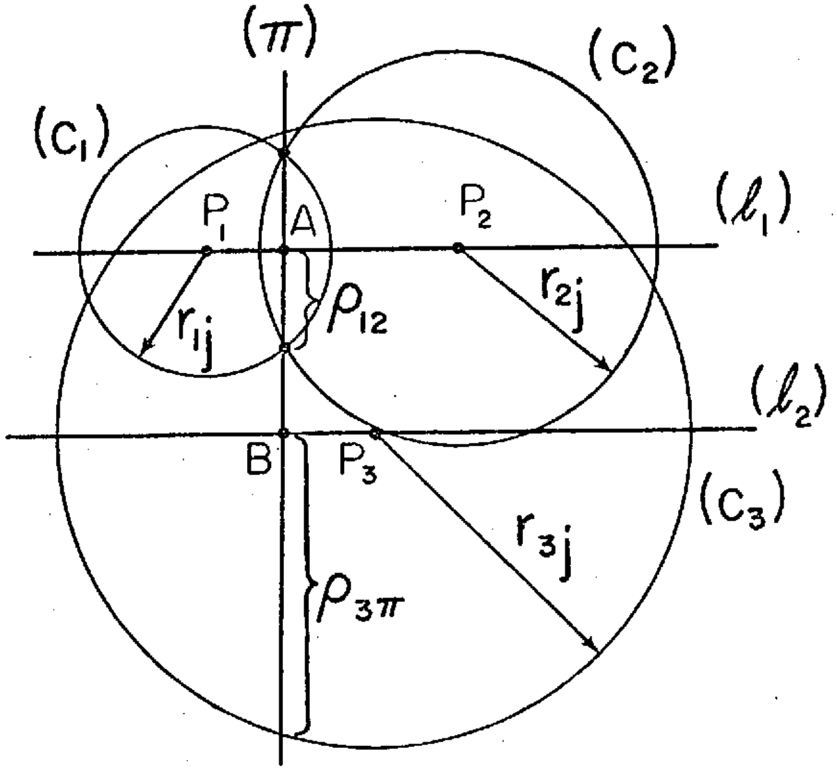
APPENDIX B

The Intersection of Three Spherical Surfaces

It is known that two spherical surfaces intersect in a circle, the plane of which is perpendicular to the line connecting the center of the spheres. Also, the intersection of a plane and a sphere is a circle with its center at the foot point of the perpendicular from the center of the sphere to the plane. The procedure to be followed is to consider the plane of the intersection of two of the three spheres, the intersection of that plane with the third sphere, and then on that plane the intersection of the two resulted circles. The points



Section of the three spheres with the plane of the centers P_1, P_2, P_3.

Figure B1.

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intersection (if they exist) of these two circles are the points of intersection of the three spherical surfaces. Since two circles intersect at most at two points (real), the set of intersection points of three spherical surfaces contains at most two points which in fact, are symmetric with respect to the plane of the centers of the three spheres. To formulate the above ideas analytically, let (C_1), (C_2), (C_3) [see Figure B1] be the three spheres with centers the ground-station points P_1, P_2, P_3 respectively. Consider the intersection of (C_1) and (C_2) and let $C_{12}(A, \rho_{12})$ be the circle of intersection, (A its center and ρ_{12} its radius) and π the plane of that circle. Let also $C_{3\pi}(B, \rho_{3\pi})$ be the circle of intersection of the plane π and the sphere (C_3). For convenience a table of the notations follows.

$C_i(P_i, r_{ij})$, $(i=1,2,3)$	- Sphere (C_i) with center P_i and radius r_{ij}.
$C_{12}(A, \rho_{12})$	- Circle of intersection of the spheres (C_1) and (C_2) with center at A and radius equal to ρ_{12}.
π	- Plane of intersection of the spheres (C_1) and (C_2).
$C_{3\pi}(B, \rho_{3\pi})$	- Circle of intersection of the plane π and the circle (C_3).
$\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_A, \bar{r}_B$	- Position vectors of the points P_1, P_2, P_3, A, B respectively.

The analytic expressions of the spherical surfaces are

$$(C_1) \ : \ ||\bar{X}||^2 - 2\bar{r}_1 \cdot \bar{X} + ||\bar{r}_1||^2 = r_{1j}^2, \tag{B1$_1$}$$

$$(C_2) \ : \ ||\bar{X}||^2 - 2\bar{r}_2 \cdot \bar{X} + ||\bar{r}_2||^2 = r_{2j}^2, \tag{B1$_2$}$$

$$(C_3) \ : \ ||\bar{X}||^2 - 2\bar{r}_3 \cdot \bar{X} + ||\bar{r}_3||^2 = r_{3j}^2, \tag{B1$_3$}$$

where vector notation is used for convenience.

Notice that

$$\bar{X} = (x,y,z)^T,\ \bar{r}_i=(x_i,y_i,z_i)^T,\ i=1,2,3,$$

$$||\bar{X}||^2= (x^2+y^2+z^2),\ ||\bar{r}_i||^2= (x_i^2+y_i^2+z_i^2),\ i=1,2,3 \tag{B2}$$

$$\text{and}\quad \bar{r}_i\cdot\bar{X} = (x_i x+y_i y+z_i z)$$

The analytic expression of the plane π (intersection of the spheres (C_1) and (C_2)) results by subtracting $(B1)_1$ from $(B1)_2$, i.e.,

$$\pi : 2(\bar{r}_2-\bar{r}_1)X = (||\bar{r}_2||^2-r_{2j}^2) - (||\bar{r}_1||^2-r_{1j}^2) \tag{B3}$$

or

$$\pi : \bar{t}\cdot\bar{X} = p, \tag{B4}$$

where

$$\bar{t} = 2(\bar{r}_2-\bar{r}_1), \tag{B5}$$

$$p = (||\bar{r}_2||^2-r_{2j}^2) - (||\bar{r}_1||^2-r_{1j}^2). \tag{B6}$$

In order to find out the position vectors of the centers and the radii of the circles $C_{12}(A,\rho_{12})$ and $C_{3\pi}(B,\rho_{3\pi})$, one considers the intersection of the plane π with (C_3) and (C_1) (or C_2) respectively. Since $BP_3//P_1P_2$ one may write for the equation of the line BP_3, (see Figure B1).

$$BP_3\equiv\ell_2 : \bar{X}=\bar{r}_3+\lambda(\bar{r}_2-\bar{r}_1)=\bar{r}_3+\lambda\bar{t},\ (\lambda= \text{a scalar}). \tag{B7}$$

By considering B as the intersection of π and $BP_3\equiv\ell_2$, its position vector results from (B4 and (B7), namely

$$\left.\begin{array}{l}\bar{X}\cdot\bar{t} = p\\ \bar{X}=\bar{r}_3+\lambda\bar{t}\end{array}\right\} \Rightarrow \left.\begin{array}{l}\bar{r}_B\cdot\bar{t} = p\\ \bar{r}_B=\bar{r}_3+\lambda_B\bar{t}\end{array}\right\} \Rightarrow \lambda_B = \frac{P=\bar{r}_3\cdot\bar{t}}{||\bar{t}||^2} = \frac{||BP_3||}{||\bar{t}||}, \tag{B8}$$

and

$$\bar{r}_B=\bar{r}_3 +[\frac{P-\bar{r}_3\cdot\bar{t}}{||t||^2}]\bar{t} \tag{B9}$$

For the radius $\rho_{3\pi}$ one observes (see Figure B1) that

$$\rho_{3\pi}^2 = r_{3j}^2 - ||BP_3||^2,$$

and by (B8)

$$\rho_{3\pi}^2 = r_{3j}^2 - [\frac{P-\bar{r}_1\cdot\bar{t}}{||t||}]^2 \ . \tag{B10}$$

Analogously for the circle $C_{12}(A,\rho_{12})$ one obtains

$$\bar{r}_A = \bar{r}_1 + [\frac{P-\bar{r}_1\cdot\bar{t}}{||t||^2}]\bar{t}, \tag{B11}$$

$$\rho_{12}^2 = r_{1j}^2 - [\frac{P-\bar{r}_1\cdot\bar{t}}{||t||}]^2 \tag{12}$$

by considering the intersection of π with (C_1), or the equivalent

$$\bar{r}_A = \bar{r}_2 + [\frac{P-\bar{r}_2\cdot\bar{t}}{||t||^2}]t, \tag{B13}$$

$$\rho_{12}^2 = r_{2j}^2 - [\frac{P-\bar{r}_2\cdot\bar{t}}{||t||}]^2, \tag{B14}$$

by considering the intersection of π with(C_2). This is all that is needed for the conditions of intersection of the three spherical surfaces. Obviously the common points to the three spherical surfaces are the common points to the two circles, $C_{12}(A,\rho_{12})$ and $C_{3\pi}(B,\rho_{3\pi})$ both of which lie on the plane π. The conditions for two circles in a plane to intersect are well known, i.e., the distance of the centers is greater than the absolute value of the difference of the radii and less than the sum of the radii. Thus the required conditions are

$$|\rho_{12}-\rho_{3\pi}| < ||\bar{r}_A-\bar{r}_B|| < \rho_{12}+\rho_{3\pi}. \tag{B15}$$

For $\bar{r}_A-\bar{r}_B$ one obtains from (B9), (B11), and (B5)

$$\bar{r}_A-\bar{r}_B = (\bar{r}_1-\bar{r}_3) - 2\frac{(\bar{r}_1-\bar{r}_3)\cdot(\bar{r}_2-\bar{r}_1)}{||\bar{r}_2-\bar{r}_1||^2}(\bar{r}_2-\bar{r}_1) \tag{B16}$$

or from (B9), (B13), and (B5) the equivalent expression

$$\bar{r}_A-\bar{r}_B = (\bar{r}_2-\bar{r}_3) - 2\frac{(\bar{r}_2-\bar{r}_3)\cdot(\bar{r}_2-\bar{r}_1)}{||\bar{r}_2-\bar{r}_1||^2}(\bar{r}_2-\bar{r}_1) \ . \tag{B17}$$

For $(\rho_{12}-\rho_{3\pi})$ and $(\rho_{12}+\rho_{3\pi})$ one uses the formulas (B12), (B10) or (B13), (B10). It is understood that one may consider first, instead of the intersection of the spheres (C_1) and (C_2) that of the spheres (C_2) and (C_3) or (C_3) and (C_1). One observes also that the intersection of the three spheres in pairs is a necessary condition but not sufficient. That is, the conditions

$$|r_{1j}-r_{2j}| < ||\bar{r}_2-\bar{r}_1|| < (r_{1j}+r_{2j}) \qquad (B18)_1$$

$$|r_{3j}-r_{2j}| < ||\bar{r}_3-\bar{r}_2|| < (r_{2j}+r_{3j}) \qquad (B18)_2$$

$$|r_{3j}-r_{1j}| < ||\bar{r}_3\ \bar{r}_1|| < (r_{3j}+r_{1j}) \qquad (B18)_3$$

are necessary for (B15) but not sufficient. After the conditions for the intersection, one may proceed to find the degenerate cases.

The left-hand side of (3) is the determinant of the matrix whose rows are the vectors $(\bar{r}_j-\bar{r}_1)$, $(\bar{r}_j-\bar{r}_2)$, $(\bar{r}_j-\bar{r}_3)$. Thus condition (3) may be interpreted as the condition for the vectors $(\bar{r}_j-\bar{r}_1)$, $(\bar{r}_j-\bar{r}_2)$, $(\bar{r}_j-\bar{r}_3)$ to be linearly independent, or equivalently <u>not</u> to be coplanar. This is equivalent to saying that the four points Q_j, P_1, P_2, P_3 are not coplanar. Since Q_j, as it was said above, does not lie in the plane of the centers P_1, P_2, P_3, the only case for the points Q_j, P_1, P_2, P_3 to be coplanar is when P_1, P_2, P_3 are collinear. Thus (3) is equivalent to stating that the centers of the three spheres are not collinear. In the case where the centers are collinear, the three spherical surfaces which have one point in common, have in common the circumference of the circle through that point whose plane is perpendicular to the line of the centers, and the center lies on that line. Thus under conditions (3) and (B15) the set of points which annul the system (2) contains two points symmetric with respect to the plane of the points P_1, P_2, P_3.

APPENDIX C

The Invariance of the Rank of the Jacobian Matrix

The invariance of the rank of the jacobian matrix is equivalent to the invariance of the jacobian determinant. To see that, consider a jacobian matrix of rank μ; that means that there exists at least one $\mu x \mu$ submatrix whose determinant is non-zero; but that determinant is the jacobian determinant of the corresponding functions; therefore the rank of the jacobian matrix is invariant under some transformations if and only if the jacobian determinant is invariant under those transformations.

Consider the set of equations

$$\begin{aligned} y_1 &= f_1(x_1,\dots,x_n) \\ y_n &= f_n(x_1,\dots,x_n). \end{aligned} \tag{C1}$$

which may be looked upon either as a transformation of coordinates or as a transformation of the points of the vector space V_n of coordinates x_i among themselves. Let $(x_1^\circ,\dots,x_n^\circ)$ be a point which corresponds to $(y_1^\circ,y_2^\circ,\dots,y_n^\circ)$. Then in the neighborhood of $(x_1^\circ,\dots,x_n^\circ)$, consider the differential space B_x which is spanned by the differentials $dx_1,\dots,dx_n$, and in the neighborhood of $(y_1^\circ,\dots,y_n^\circ)$ the differential space B_y which is spanned by $dy_1,\dots,dy_n$. The jacobian matrix is the matrix of the linear transformation under which the differential space B_x goes into the differential space B_y, i.e.,

$$\begin{bmatrix} dy_1 \\ \cdot \\ \cdot \\ \cdot \\ dy_n \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1},\dots,\frac{\partial f_1}{\partial x_n} \\ \dots\ \dots\ \dots \\ \frac{\partial f_n}{\partial f_1},\dots,\frac{\partial f_n}{\partial f_n} \end{bmatrix} \begin{bmatrix} dx_1 \\ \cdot \\ \cdot \\ \cdot \\ dx_n \end{bmatrix}, \tag{C2}$$

or

$$dy = J_M \cdot dx. \tag{C3}$$

Suppose that $dx_1, dx_2, \ldots, dx_n$ are independent from each other, that is they constitute a basis for the differential space B_x, and this is the case for $dy_1, \ldots, dy_n$ in B_y.

Consider the affine transformation

$$Z = Ax+\beta, \quad |A| \neq 0, \tag{C4}$$

where A is a constant nxn matrix, β is a constant n-vector, $x=(x_1,\ldots,x_n)^T$ and $z=(z_1,\ldots,z_n)^T$ the vector onto which x goes under the transformation (C4). Then the jacobian matrix J_M in (C3) is transformed as follows:

$$\frac{\partial f_i}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k} \frac{\partial z_k}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k} a_{kj} \tag{C5}$$

where a_{kj} is the (k,j) element of the matrix A. Then one may write in matrix notation

$$J_M(x) = J_M(z)A. \tag{C6}$$

By taking the determinants of both sides one has

$$|J_M(x)| = |J_M(z)| \cdot |A| = |A| \cdot |J_M(z)|. \tag{C7}$$

Hence $|J_M(x)|$ is a relative invariant of weight* one, under the transformation (C4). Differentiation of (C4) yields

$$dz = Adx \text{ or } dx = A^{-1}dz, \tag{C8}$$

and then substitution of (C6) and (C8) into (C3)

$$dy = J_M(x)dx = J_M(z)A\,A^{-1}dz = J_M(z)dz. \tag{C9}$$

The last result shows that $J_M(x)dx$ is an invariant, under the transformation (C4).

*It is recalled that the weight is rendered by the exponent of the determinant of the matrix of the linear operator. If the exponent is zero then one has an absolute invariant, otherwise a relative one.

Since the jacobian determinant is invariant under the affine transformations, the rank of the jacobian matrix is invariant also, under the same transformations.

APPENDIX D

Pivotal Condensation ([5], pp. 45-48)

By condensation of a determinant it is meant the reduction of its order in a systematic way. There are various methods of condensation, from which the method of pivotal condensation is presented here. Since pivotal condensation corresponds exactly to the process of successive elimination which is used in the solution of simultaneous linear equations, it seems to be easier to introduce the method from that side.

Consider for example the following set of simultaneous equations:

$$\begin{aligned} a_1x_1+b_1x_2+c_1x_3+d_1x_4 &= h_1 \\ a_2x_1+b_2x_2+c_2x_3+d_2x_4 &= h_2 \\ a_3x_1+b_3x_2+c_3x_3+d_3x_4 &= h_3 \\ a_4x_1+b_4x_2+c_4x_3+d_4x_4 &= h_4, \end{aligned} \tag{D1}$$

and try to eliminate the variable x_1. To do that one may proceed as follows: by multiplying all the equations except the first by a_1 and then performing the operations

2^{nd} eqt. $- a_2(1^{st}$ eqt)

3^{d} eqt. $- a_3(1^{st}$ eqt)

4^{th} eqt. $- a_4(1^{st}$ eqt)

to obtain

$$\begin{array}{lllll} a_1x_1 & + \quad b_1x_2+ & c_1x_3+ & d_1x_4 & = h_1 \\ 0 & + (a_1b_2-b_1a_2)x_2+ & (a_1c_2-c_1a_2)x_3+ & (a_1d_2-d_1a_2)x_4 & = h_2-a_2h_1 \\ 0 & + (a_1b_3-b_1a_3)x_2+ & (a_1c_3-c_1a_3)x_3+ & (a_1d_3-d_1a_3)x_4 & = h_3-a_3h_1 \\ 0 & + (a_1b_4-b_1a_4)x_2+ & (a_1c_4-c_1a_4)x_3+ & (a_1d_4-d_1a_4)x_4 & = h_4-a_4h_1 \end{array} \tag{D2}$$

Now consider the determinant of the coefficient matrix, and observe the above operations for the elimination of x_1, on that determinant.

$$|A| = |a_1b_2c_3d_4| = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \ (*), \tag{D3}$$

$$a_1^3|A| = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_1a_2 & a_1b_2 & a_1c_2 & a_1d_2 \\ a_1a_3 & a_1b_3 & a_1c_3 & a_1d_3 \\ a_1a_4 & a_1b_4 & a_1c_4 & a_1d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & |a_1b_2| & |a_1c_2| & |a_1d_2| \\ 0 & |a_1b_3| & |a_1c_3| & |a_1d_3| \\ 0 & |a_1b_4| & |a_1c_4| & |a_1d_4| \end{vmatrix} \Rightarrow$$

$$|A| = \frac{1}{a_1^2} \begin{vmatrix} |a_1b_2| & |a_1c_2| & |a_1d_2| \\ |a_1b_3| & |a_1c_3| & |a_1d_3| \\ |a_1b_4| & |a_1c_4| & |a_1d_4| \end{vmatrix}. \tag{D4}$$

Thus the original determinant $|A|$ has been reduced to one of third order. In general a determinant of order n is equal to $1/a_1^{n-2}$ times a determinant of order n-1, the elements of which are the minors of $|A|$ of order 2 having a_1 as leading element placed in proper priority of rows and columns. Thus the minor belonging to the 1^{st} and $(i+1)^{th}$ rows, 1^{st} and $(j+1)^{st}$ columns of $|A|$ will be the $(i,j)^{th}$ element of the condensed determinant. The element a_1 is called <u>pivotal element</u> or <u>pivot</u>. The pivot need not be the leading element, which may be zero. Any convenient non-zero element may be taken as pivot, for one could make any element a

*The representation of a determinant as a row of its diagonal elements is very convenient, in this sort of manipulations, and it is adopted in the present appendix.

leading element by permuting rows and columns. The pivots which are leading elements are more convenient. Next to the leading pivots in convenience come the diagonal elements.

Not only single elements may be used as pivots, but any non-zero minor. Principal minors are preferable. The procedure is the same as that of a single element pivot. Thus a determinant $|A|$ of order n which is to be condensed by using as pivot a principal minor of order m $(m<n)$, is equal to $(\Delta_m)^{-(n-m-1)}$ times a determinant of order $(n-m)$ the elements of which are the minors of $|A|$ of order $(m+1)$ having Δ_m as leading principal minor, placed in proper priority of rows and columns.

Before closing this appendix, a series of important identities are given.

$$|a_{11}a_{22}a_{33}| = a_{11}^{-1} \begin{Vmatrix} |a_{11}a_{22}| & |a_{11}a_{23}| \\ |a_{11}a_{32}| & |a_{11}a_{33}| \end{Vmatrix} \tag{D5}$$

$$|a_{11}a_{22}a_{33}a_{44}| = a_{11}^{-1} \begin{Vmatrix} |a_{11}a_{22}| & |a_{11}a_{23}| & |a_{11}a_{24}| \\ |a_{11}a_{32}| & |a_{11}a_{33}| & |a_{11}a_{34}| \\ |a_{11}a_{42}| & |a_{11}a_{43}| & |a_{11}a_{44}| \end{Vmatrix} \tag{D6}$$

$$|a_{11}a_{22}a_{33}a_{44}| = |a_{11}a_{22}|^{-2} \begin{Vmatrix} |a_{11}a_{22}a_{33}| & |a_{11}a_{22}a_{34}| \\ |a_{11}a_{22}a_{43}| & |a_{11}a_{22}a_{44}| \end{Vmatrix} \tag{D7}$$

$$|a_{11}a_{22}a_{33}a_{44}a_{55}| = |a_{11}a_{22}a_{33}|^{-1} \begin{Vmatrix} |a_{11}a_{22}a_{33}a_{44}| & |a_{11}a_{22}a_{33}a_{44}| \\ |a_{11}a_{22}a_{33}a_{54}| & |a_{11}a_{22}a_{33}a_{55}| \end{Vmatrix} \tag{D8}$$

$$
|a_{11}\cdots a_{nn}| = |a_{11}\cdots a_{mm}|^{-(n-m-1)}
\begin{vmatrix}
|a_{11}\cdots a_{mm}a_{m+1,m+1}| & |a_{11}\cdots a_{mm}a_{m+1,m+2}| & \cdots & |a_{11}\cdots a_{mm}a_{m+1,n}| \\
|a_{11}\cdots a_{mm}a_{m+2,m+1}| & |a_{11}\cdots a_{mm}a_{m+2,m+2}| & \cdots & |a_{11}\cdots a_{mm}a_{m+2,n}| \\
\cdots\cdots & \cdots\cdots & & \cdots\cdots \\
|a_{11}\cdots a_{mm}a_{n,m+1}| & |a_{11}\cdots a_{mm}a_{n,m+2}| & \cdots & |a_{11}\cdots a_{mm}a_{n,n}|
\end{vmatrix}
\quad \text{(D9)}
$$

APPENDIX E

Condensation of the Jacobian Matrix For Four Stations in Range Networks

i) Pivotal Condensation

Let $P_1 \equiv (x_1, y_1, z_1)$, $P_2 \equiv (x_2, y_2, z_2)$, $P_3 \equiv (x_3, y_3, z_3)$, $P_4 \equiv (x_4, y_4, z_4)$ be four simultaneously observing ground stations, and $Q_j \equiv (X_j, Y_j, Z_j)$, $(j=1,2,\ldots,m)$ be a set of satellite-position points observed by those stations. Then the functional relationships on which the statistical model is based, are

$$f_{ij} \equiv (X_j - x_i)^2 + (Y_j - y_i)^2 + (Z_j - z_i)^2 = r_{ij}^2$$

$$i = 1,2,3,4 \qquad \text{(E1)}$$

$$j = 1,2,\ldots,m, m \geq 6$$

(see Sections 1.2 and 1.3).

This system contains

$$\nu = 4m \geq 24 \text{ equations,} \qquad \text{(E2)}$$

and

$$3m + 3x4 = 3(m+4) \text{ unknowns,} \qquad \text{(E3)}$$

among which six are to be constrained, so that there are

$$\mu = 3(m+2) \text{ actual unknowns.} \qquad \text{(E4)}$$

Nevertheless while forming the jacobian matrix, (J_M) for the system (E1), no unknown variable is considered constrained. Thus the jacobian matrix of the system (E1) is of dimensions 4mx(3m+4). Since (E1) is cyclicly symmetric with respect to X_j, Y_j, Z_j and x_i, y_i, z_i, while writing the explicit form of the jacobian matrix only the columns for X_j and x_i will be written down for simplicity. The columns for Y_jZ_j and y_ix_i will be understood to follow immediately those of X_j and x_i respectively. Thus, one has for the jacobian matrix:

(i,j)	X_1	...	X_j	...	X_m	x_1	x_2	x_3	x_4
(1,1)	$2(X_1-x_1)$	0...	0	...	0	$-2(X_1-x_1)$	0	0	0
...	...	...	...	...	...	...	...	...	...
1,j	0	...0	$2(X_j-x_1)$	0...	0	$-2(X_j-x_1)$	0	0	0
...	...	...	...	...	...	...	...	...	...
(1,m)	0	...	0	...0	$2(X_m-x_1)$	$-2(X_m-x_1)$	0	0	0
(2,1)	$2(X_1-x_2)$	0...	0	...	0	0	$-2(X_1-x_2)$	0	0
...	...	...	...	...	...	...	...	...	...
(2,j)	0	...0	$2(X_j-x_2)$	0...	0	0	$-2(X_j-x_2)$	0	0
...	...	...	...	...	...	...	...	...	...
(2,m)	0	...	0	...0	$2(X_m-x_2)$	0	$-2(X_m-x_2)$	0	0
(3,1)	$2(X_1-x_3)$	0...	0	...	0	0	0	$-2(X_1-x_3)$	0
...	...	...	...	...	...	...	...	...	...
(3,j)	0	...0	$2(X_j-x_3)$	0...	0	0	0	$-2(X_j-x_3)$	0
...	...	...	...	...	...	...	...	...	...
(3,m)	0	...	0	...0	$2(X_m-x_3)$	0	0	$-2(X_j-x_3)$	0
(4,1)	$2(X_1-x_4)$	0...	0	...	0	0	0	0	$-2(X_1-x_4)$
...	...	...	...	...	...	...	...	...	...
(4,j)	0	...0	$2(X_j-x_4)$	0...	0	0	0	0	$-2(X_j-x_4)$
...	...	...	...	...	...	...	...	...	...
(4,m)	0	...	0	...0	$2(X_m-x_4)$	0	0	0	$-2(X_m-x_4)$

Table E1. The Jacobian Matrix for the Case of Four Stations
Note : Only the x-columns appear for brevity.

According to what was said in Section 1.3, the least squares solution is unique iff

$$\text{rank } J_M = \mu = 3(m+2) \tag{E5}$$

while for

$$\text{rank } J_M < \mu = 3(m+2)$$

one has the determinantal loci of the configuration. Therefore in order to find out the determinantal loci, one has to consider the determinants of all $\mu x \mu$ submatrices of J_M, where the value of $\mu = 3(m+2)$ is determined through the value of m, which is given from the equation

$$rm = 3(m+2)\text{ , i.e., } m=6 \text{ .} \tag{E6}$$

Thus

$$\mu = 3(m+2) = 24 \text{ .} \tag{E7}$$

It is reminded that m=6 is the minimum number of events for a unique solution in the case of four stations, (see Section 1.1.3 relations [6]).

At this point one is compelled to constrain six of the unknown variables, in order to be able to pick up a 24x24 submatrix. However it is desirable to arrive at general expressions, before one specifies the coordinates to be constrained. In the following the method of pivotal condensation will be used to simplify the determinants of those $\mu x \mu = 24x24$ submatrices. Working out that method with a specific set of admissible constraints a result will be obtained the symmetry of which will lead to a general expression.

Assume that $x_1, y_1, z_1, y_2, z_2, z_3$ are constrained and consider, without loss of generality, the submatrix of the $\mu(=24)$ first rows of the jacobian matrix. In order to get this submatrix, put m=6 and ignore columns $x_1, y_1, z_1, y_2, z_2, z_3$ in Table 1. of the jacobian matrix. Thus the following determinant results, where the factor 2 has been dropped from each entry:

	1	4	7	10	13	16	19	20	22
	X_1	X_2	X_3	X_4	X_5	X_6	x_2	x_3	x_4
1	X_1-x_1	0	0	0	0	0	0	0	0
2	0	X_2-x_1	0	0	0	0	0	0	0
3	0	0	X_3-x_1	0	0	0	0	0	0
4	0	0	0	X_4-x_1	0	0	0	0	0
5	0	0	0	0	X_5-x_1	0	0	0	0
6	0	0	0	0	0	X_6-x_1	0	0	0
7	X_1-x_2	0	0	0	0	0	$-(X_1-x_2)$	0	0
8	0	X_2-x_2	0	0	0	0	$-(X_2-x_2)$	0	0
9	0	0	X_3-x_2	0	0	0	$-(X_3-x_2)$	0	0
10	0	0	0	X_4-x_2	0	0	$-(X_4-x_2)$	0	0
11	0	0	0	0	X_5-x_2	0	$-(X_5-x_2)$	0	0
12	0	0	0	0	0	X_6-x_2	$-(X_6-x_2)$	0	0
13	X_1-x_3	0	0	0	0	0	0	$-(X_1-x_3)$	0
14	0	X_2-x_3	0	0	0	0	0	$-(X_2-x_3)$	0
15	0	0	X_3-x_3	0	0	0	0	$-(X_3-x_3)$	0
16	0	0	0	X_4-x_3	0	0	0	$-(X_4-x_3)$	0
17	0	0	0	0	X_5-x_3	0	0	$-(X_5-x_3)$	0
18	0	0	0	0	0	X_6-x_3	0	$-(X_6-x_3)$	0
19	X_1-x_4	0	0	0	0	0	0	0	$-(X_1-x_4)$
20	0	X_2-x_4	0	0	0	0	0	0	$-(X_2-x_4)$
21	0	0	X_3-x_4	0	0	0	0	0	$-(X_3-x_4)$
22	0	0	0	X_4-x_4	0	0	0	0	$-(X_4-x_4)$
23	0	0	0	0	X_5-x_4	0	0	0	$-(X_5-x_4)$
24	0	0	0	0	0	X_6-x_4	0	0	$-(X_6-x_4)$

Table E2. Determinant of a 24x24 Submatrix of the Jacobian Matrix, (see Table E1).

This determinant should be expanded and its zeroes be searched for the determinantal loci. There are many methods of expanding a determinant, the most appropriate one here being that of pivotal condensation, (see Appendix D).

Take the principal minor of the first eighteen rows as the leading pivot (assumed non-zero) and let it be called $|\Delta_{18}|$. Also interpret the symbol.

$$|\Delta_{18},(\rho,\lambda)|$$

as the principal minor which results by juxtaposing the ρ^{th}-row and λ^{th}-column with the principal minor $|\Delta_{18}|$. Then one has for the above determinant $|\Delta|$:

$$|\Delta| = \frac{1}{|\Delta_{18}|^5} \begin{vmatrix} |\Delta_{18},(19,19)|, & |\Delta_{18},(19,20)|, & |\Delta_{18},(19,21)|, & |\Delta_{18},(19,22)|, & |\Delta_{18},(19,23)|, & |\Delta_{18},(19,24)| \\ |\Delta_{18},(20,19)|, & |\Delta_{18},(20,20)|, & |\Delta_{18},(20,21)|, & |\Delta_{18},(20,22)|, & |\Delta_{18},(20,23)|, & |\Delta_{18},(20,24)| \\ |\Delta_{18},(21,19)|, & |\Delta_{18},(21,20)|, & |\Delta_{18},(21,21)|, & |\Delta_{18},(21,22)|, & |\Delta_{18},(21,23)|, & |\Delta_{18},(21,24)| \\ |\Delta_{18},(22,19)|, & |\Delta_{18},(22,20)|, & |\Delta_{18},(22,21)|, & |\Delta_{18},(22,22)|, & |\Delta_{18},(22,23)|, & |\Delta_{18},(22,24)| \\ |\Delta_{18},(23,19)|, & |\Delta_{18},(23,20)|, & |\Delta_{18},(23,21)|, & |\Delta_{18},(23,22)|, & |\Delta_{18},(23,23)|, & |\Delta_{18},(23,24)| \\ |\Delta_{18},(24,19)|, & |\Delta_{18},(24,20)|, & |\Delta_{18},(24,21)|, & |\Delta_{18},(24,22)|, & |\Delta_{18},(24,23)|, & |\Delta_{18},(24,24)| \end{vmatrix} \quad \text{(E8)}$$

This is a compound determinant (i.e., a determinant having determinant elements). For its explicit form the determinant elements are expanded by Laplacian expansion:

(a) Laplacian expansion of $|\Delta_{18}|$.

Expanding $|\Delta_{18}|$ according to its first three columns there is only one non-zero 3x3-minor which being multiplied by its cofactor gives the only term of the expansion. The same is repeated with the cofactor and the process continues similarly, until one obtains:

$$|\Delta_{18}| = (-1)^{\sigma}\Delta_4^1\cdot\Delta_4^2\cdot\Delta_4^3\cdot\Delta_4^4\cdot\Delta_4^5\cdot\Delta_4^6 = -\prod_{j=1}^{6} \Delta_4^j, \tag{E9}$$

where

$$\sigma=5(1+2+3)+(1+7+13)+(1+6+11)+(1+5+9)+(1+4+7)+(1+3+5) = 105$$

and

$$\Delta_4^j = \begin{vmatrix} X_j-x_1 & Y_j-y_1 & Z_j-z_1 \\ X_j-x_2 & Y_j-y_2 & Z_j-z_2 \\ X_j-x_3 & Y_j-y_3 & Z_j-z_3 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}. \tag{E10}$$

It is appropriate at this place to introduce the determinant

$$\Delta^j = \begin{vmatrix} X_j-x_1 & Y_j-y_1 & Z_j-z_1 & 1 \\ X_j-x_2 & Y_j-y_2 & Z_j-z_2 & 1 \\ X_j-x_3 & Y_j-y_3 & Z_j-z_3 & 1 \\ X_j-x_4 & Y_j-y_4 & Z_j-z_4 & 1 \end{vmatrix}. \tag{E11}$$

After some row and column manipulations Δ^j becomes

$$\Delta^j = -\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \tag{E12}$$

Notice that Δ^j is independent of the satellite coordinates. Expanding Δ^j according to the elements of the last column one has

$$\Delta^j = \Delta_1^j - \Delta_2^j + \Delta_3^j - \Delta_4^j \,, \qquad \text{(E13)}$$

where

$$\Delta_1^j = \begin{vmatrix} X_j-x_2 & Y_j-y_2 & Z_j-z_2 \\ X_j-x_3 & Y_j-y_3 & Z_j-z_3 \\ X_j-x_4 & Y_j-y_4 & Z_j-z_4 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

$$\Delta_2^j = \begin{vmatrix} X_j-x_1 & Y_j-y_1 & Z_j-z_1 \\ X_j-x_3 & Y_j-y_3 & Z_j-z_3 \\ X_j-x_4 & Y_j-y_4 & Z_j-z_4 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \qquad \text{(E14)}$$

$$\Delta_3^j = \begin{vmatrix} X_j-x_1 & Y_j-y_1 & Z_j-z_1 \\ X_j-x_2 & Y\ -y_2 & Z_j-z_2 \\ X_j-x_4 & Y_j-y_4 & Z_j-z_4 \end{vmatrix} = \begin{vmatrix} X_j & Y_j & Z_j & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ z_4 & y_4 & z_4 & 1 \end{vmatrix}$$

and Δ_4^j is given by (E10).

(b) <u>Laplacian expansion of a determinant-element of $|\Delta|$, say $|\Delta_{18},(19,19)|$</u>

Expanding this determinant according to the last column, only one term is non-zero, namely that which corresponds to $-(X_1-x_2)$. The other five terms are obviously zero. To see that, take the minor of $-(X_2-x_2)$, and try to obtain its Laplacian expansion according to the columns 4,5,6; it is zero, for there is not any non-zero 3x3-minor there. The minor of $-(X_1-x_2)$ is of the same form with $|\Delta_{18}|$. As a matter of fact, it results from $|\Delta_{18}|$ within sign by deleting the 7-th row and introducing the 19-th one; in this case the sign changes for the 19-th row get the place of the 7-th row by 11 row-interchanges. Therefore working as before one obtains:

$$|\Delta_{18},(19,19)| = -(X_1-x_2)\Delta_2^1\cdot\Delta_4^2\cdot\Delta_4^3\cdot\Delta_4^4\cdot\Delta_4^5\cdot\Delta_4^6 \tag{E15}$$

or

$$|\Delta_{18},(19,19)| = -(X_1-x_2)\,\frac{\Delta_2^1}{\Delta_4^1}\prod_{j=1}^{6}\Delta_4^j = +(X_1-x_2)\frac{\Delta_2^1}{\Delta_4^1}\,|\Delta_{19}|. \tag{E16}$$

In an exactly similar way one obtains the expansions of all the determinant-elements of $|\Delta|$. Dividing each row of the determinant $|\Delta|$ by $|\Delta_{18}|$, and taking out of each row j the common factor $1/\Delta_4^j$ one has

$$|\Delta| = \begin{vmatrix} -(X_1-x_2)\Delta_2^1, & (X_1-x_3)\Delta_3^1, & (Y_1-y_3)\Delta_3^1, & -(X_1-x_4)\Delta_4^1, & -(Y_1-y_4)\Delta_4^1, & -(Z_1-z_4)\Delta_4^1 \\ -(X_2-x_2)\Delta_2^2, & (X_2-x_3)\Delta_3^2, & (Y_2-y_3)\Delta_3^2, & -(X_2-x_4)\Delta_4^2, & -(Y_2-y_4)\Delta_4^2, & -(Z_2-z_4)\Delta_4^2 \\ -(X_3-x_2)\Delta_2^3, & (X_3-x_3)\Delta_3^3, & (Y_3-y_3)\Delta_3^3, & -(X_3-x_4)\Delta_4^3, & -(Y_3-y_4)\Delta_4^3, & -(Z_3-z_4)\Delta_4^3 \\ -(X_4-x_2)\Delta_2^4, & (X_4-x_3)\Delta_3^4, & (Y_4-y_3)\Delta_3^4, & -(X_4-x_4)\Delta_4^4, & -(Y_4-y_4)\Delta_4^4, & -(Z_4-z_4)\Delta_4^4 \\ -(X_5-x_2)\Delta_2^5, & (X_5-x_3)\Delta_3^5, & (Y_5-y_3)\Delta_3^5, & -(X_5-x_4)\Delta_4^5, & -(Y_5-y_4)\Delta_4^5, & -(Z_5-z_4)\Delta_4^5 \\ -(X_6-x_2)\Delta_2^6, & (X_6-x_3)\Delta_3^6, & (Y_6-y_3)\Delta_3^6, & -(X_6-x_4)\Delta_4^6, & -(Y_6-y_4)\Delta_4^6, & -(Z_6-z_4)\Delta_4^6 \end{vmatrix} \tag{E17}$$

The symmetry of the above expression suggests the general form one is looking for, where the constrained coordinates are not specified. That general expression is given below, and it is a sort of condensed jacobian matrix. Each time a set of admissible constraints is selected, one has to delete the corresponding columns from the general array which follows.

x_1	y_1	z_1	x_2	y_2	z_2	x_3	y_3	z_3	x_4	y_4	z_4
+	+	+	-	-	-	+	+	+	-	-	-
$(X_j-x_1)\Delta_1^j$	$(Y_j-y_1)\Delta_1^j$	$(Z_j-z_1)\Delta_1^j$	$(X_j-x_2)\Delta_2^j$	$(Y_j-y_2)\Delta_2^j$	$(Z_j-z_2)\Delta_2^j$	$(X_j-x_3)\Delta_3^j$	$(Y_j-y_3)\Delta_3^j$	$(Z_j-z_3)\Delta_3^j$	$(X_j-x_4)\Delta_4^j$	$(Y_j-y_4)\Delta_4^j$	$(Z_j-z_4)\Delta_4^j$

$j=1,2,3,\ldots,m$

Table E3. The Condensed Jacobian Matrix for the Case of Four Ground Stations

For example the expression (E17) of the determinant $|\Delta|$ is obtained immediately from the condensed jacobian matrix by deleting the columns of the constrained coordinates $x_1, y_1, z_1, y_2, z_2, z_3$ and selecting for the index j the values 1,2,3,4,5,6 out of the m.

ii) By the method of elimination

It was said in Appendix D that pivotal condensation corresponds exactly to the process of successive elimination which is used in the solution of simultaneous equations. This fact will be exploited to obtain the condensed jacobian matrix by using a technique different from the previous one. Consider the differential form of the equations (E1).

$$2(X_j-x_i)(dX_j-dx_i)+2(Y_j-y_i)(dY_j-dy_i)+2(Z_j-z_i)(dZ_j-dz_i) = 2r_{ij}dr_{ij}. \quad \text{(E18)}$$

This is a linear mapping of the differential space spanned by the differentials $(dX_j, dY_j, dZ_j, dx_i, dy_i, dz_i)$ into the differential space spanned by dr_{ij}, ($i=1,2,3,4$ and $j=1,2,\ldots,m$). Since one is interested here in the rank of the matrix of the above linear operator, he may consider the kernel or null space of that operator, i.e., he may take

$$(X_j-x_i)(dX_j-dx_i)+(Y_j-y_i)(dY_j-dy_i)+(Z_j-z_i)(dZ_j-dz_i) = 0, \quad \text{(E19)}$$

to obtain all the information about the rank.

One may group the 4m equations (E19) in m groups of four, of the form

$$
\begin{aligned}
(X_j-x_1)(dX_j-dx_1)+(Y_j-y_1)(dY_j-dy_1)+(Z_j-z_1)(dZ_j-dz_1) &= 0\\
(X_j-x_2)(dX_j-dx_2)+(Y_j-y_2)(dY_j-dy_2)+(Z_j-z_2)(dZ_j-dz_2) &= 0\\
(X_j-x_3)(dX_j-dx_3)+(Y_j-y_3)(dY_j-dy_3)+(Z_j-z_3)(dZ_j-dz_3) &= 0\\
(X_j-x_4)(dX_j-dx_4)+(Y_j-y_4)(dY_j-dy_4)+(Z_j-z_4)(dZ_j-dz_4) &= 0.
\end{aligned}
\tag{E20}
$$

From the above four simultaneous linear homogeneous equations one may eliminate the three differentials dX_j, dY_j, dZ_j. The resultant of the elimination is a linear homogenous equation of the form

$$a_1dx_1+a_2dy_1+a_3dz_1+a_4dx_2+a_5dy_2+a_6dz_2+a_7dx_3+a_8dy_3+a_9dz_3+a_{10}dx_4+a_{11}dy_4+a_{12}dz_4=0,$$

where the coefficients $a_i, (i=1,\dots,12)$ are the entries of the j-th row of the condensed jacobian matrix as it is given in Table E3. The elimination of dX_j, dY_j, dZ_j is carried out formally; one takes three of the equations (E20), solves them for dX_j, dY_j, dZ_j and then substitutes in the fourth one to obtain the resultant equation.

This method is followed in (Blaha [1971]) and (Rinner [1966]) with the difference that the constrained coordinates are specified from the beginning, and (E20) becomes

$$
\begin{aligned}
&X_jdX_j + Y_jdY_j + Z_jdZ_j = 0\\
&(X_j-x_2)(dX_j-dx_2) + Y_jdY_j + Z_jdZ_j = 0\\
&(X_j-x_3)(dX_j-dx_3) + (Y_j-y_3)(dY_j-dy_3) + Z_jdZ_j = 0\\
&(X_j-x_4)(dX_j-dx_4) + (Y_j-y_4)(dY_j-dy_4) + (Z_j-z_4)(dZ_j-dz_4) = 0
\end{aligned}
\tag{E21}
$$

under the constraints $x_1=y_1=z_1=y_2=z_2=z_3=0$.

One observes that symmetry is gone in (E21), and that fact has many consequences for the whole treatment. It is not advisable for a theoretical treatment to destroy symmetry of expressions, for such an act deprives one of flexibility in the manipulations.

The expressions (E14) for the particular set of constraints: $x_1=y_1=z_1=y_2=z_2=z_3=0$, become :

$$\Delta_1^j = Z_j \begin{vmatrix} x_2 & 0 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - Z_4 \begin{vmatrix} X_j & Y_j & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta_2^j = Z_j \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + Z_4 \begin{vmatrix} X_j & Y_j \\ x_3 & y_3 \end{vmatrix} \tag{E22}$$

$$\Delta_3^j = Z_j \; x_2 y_4 - z_4 x_2 y_j$$

$$\Delta_4^j = Z_j \; x_2 y_3 .$$

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APPENDIX F

The Complete Invariant System Associated With a Set of Geometric Points

The purpose of this appendix is to quote some applications of the theory of invariants to geometry, which are of interest in this work. The discussion is within the frame of affine geometry. At first the definitions of fundamental invariants will be given.

(i) One Dimensional Space

Consider two points P_1 and P_2 with affine coordinates $(x_1,1)$ and $(x_2,1)$ respectively. Take the determinant

$$\Delta_{12} = \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = x_1 - x_2. \tag{F1}$$

This two-rowed determinant which is formed from the coordinates of the two points is a fundamental invariant in one-dimensional space. Actually Δ_{12} is an expression of the distance between the two points.

(ii) Two Dimensional Space

Consider three points P_1,P_2,P_3 with affine coordinates $(x_1,y_1,1)$, $(x_2,y_2,1)$, $(x_3,y_3,1)$ respectively. The three-rowed determinant which is formed from these coordinates, i.e.,

$$\Delta_{123} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} , \tag{F2}$$

is a fundamental invariant in two dimensional space. It is well known that Δ_{123} is twice the area of the triangle $(P_1P_2P_3)$.

(iii) Three Dimensional Space

Consider four points P_1, P_2, P_3, P_4 with affine coordinates $(x_1\ y_1, z_1, 1)$, $(x_2, y_2, z_2, 1)$ $(x_3, y_3, z_3, 1)$, $(x_4, y_4, z_4, 1)$ respectively. The four-rowed determinant which is formed from these coordinates, i.e.,

$$\Delta_{1234} = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}, \tag{F3}$$

is a fundamental invariant in three dimensional space. It is known that Δ_{1234} is six times the volume of the tetrahedron $(P_1P_2P_3P_4)$.

The theory of invariants proves:

i) Given a finite number of points on a straight line, $[P_1 \equiv (x_1, 1), \ldots P_n(x_n, 1)]$, the complete invariant system associated with these points is the totality of the two-rowed determinants.

$$\Delta_{ij} = \begin{vmatrix} x_i & 1 \\ x_j & 1 \end{vmatrix} \qquad \begin{matrix} i=1,2,\ldots,n \\ j=1,2,\ldots,n. \end{matrix} \tag{F4}$$

That means that any invariant associated with these points can be expressed as a polynomial in the Δ_{ij}.

ii) Given a finite set of points in a plane $[P_1 \equiv (x_1, y_1, 1), \ldots, P_n \equiv (x_n, y_n, 1)]$, the complete invariant system associated with these points is the totality of the three-rowed determinants.

$$\Delta_{ijk} = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} \qquad \begin{matrix} i=1,2,\ldots,n \\ j=1,2,\ldots,n \\ k=1,2,\ldots,n, \end{matrix} \tag{F5}$$

iii) Given a finite set of points in the three dimensional space

$P_1 \equiv (x_1, y_1, z_1, 1), \ \ldots, \ P_n \equiv (x_n, y_n, z_n, 1)$, the complete invariant system associated with these points is the totality of the four-rowed determinants

$$\Delta_{ijk\ell} = \begin{vmatrix} x_i & y_i & z_i & 1 \\ x_j & y_j & z_j & 1 \\ x_k & y_k & z_k & 1 \\ x_\ell & y_\ell & z_\ell & 1 \end{vmatrix} , \quad \begin{array}{l} i=1,2,\ldots,n \\ j=1,2,\ldots,n \\ k=1,2,\ldots,n \\ \ell=1,2,\ldots,n. \end{array} \tag{F6}$$

All these fundamental invariants are relative invariants of weight one under affine transformations. Consider for example the determinant (F3) and the affine transformation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} . \tag{F7}$$

Then one has

$$\Delta'_{1234} = \begin{vmatrix} x'_1 & y'_1 & z'_1 & 1 \\ x'_2 & y'_2 & z'_2 & 1 \\ x'_3 & y'_3 & z'_3 & 1 \\ x'_4 & y'_4 & z'_4 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} |A|$$

or

$$\Delta'_{1234} = |A| \; \Delta_{1234}. \tag{F8}$$

The last relation shows that Δ_{1234} is a relative invariant of weight one.

In the problem of determinantal loci one has a finite set of points in the three dimensional space, namely n station-position points and m satellite-position points. Therefore any invariant under affine transformations, which is a rational integral function of the coordinates of these points, must be expressible as a polynomial in the fundamental invariants (F6). As an example of this take the jacobian matrix for the range observations from four ground stations. (see Table E3, Appendix E). It has been proved in Appendix C that the determinants of its square submatrices are invariant under affine transformations. Take a square submatrix of that matrix, say that of the first twelve rows. That is a rational integral function of the coordinates and invariant under affine transformations. Therefore it must be expressible as a polynomial in the fundamental invariants (F6). This polynomial can be obtained in three embedded steps of Laplace expansion:

1st step - expand the 12-order determinant in Laplace expansion according to the first three columns;

2nd step - expand the 9-order determinant of each term in the 1st step in Laplace expansion according to its first three columns, and

3rd step - expand the 6-order determinant of each term in the 2nd step in Laplace expansion according to its first three columns.

Then one obtains

$$\Delta_{j_1,\ldots\ldots,j_{12}} = \sum_{\rho_1(\rho_2(\rho_3))} (-1)^{\sigma_1+\sigma_2+\sigma_3} \Delta_1^{j_1}\Delta_1^{j_2}\Delta_1^{j_3}\Delta_2^{j_4}\Delta_2^{j_5}\Delta_2^{j_6}\Delta_3^{j_7}\Delta_3^{j_8}\Delta_3^{j_9}\Delta_4^{j_{10}}\Delta_4^{j_{11}}\Delta_4^{j_{12}}\mathfrak{D}_1^{j_1,j_2,j_3}\mathfrak{D}_2^{j_4,j_5,j_6}\mathfrak{D}_3^{j_7,j_8,j_9}\mathfrak{D}_4^{j_{10},j_{11},j_{12}} \tag{F9}$$

where $\sigma_1,\sigma_2,\sigma_3$ result from the above Laplace expansions and ρ_1,ρ_2,ρ_3 are selections of row indices, of these expansions. Δ_i^j are given in (E14) of Appendix E, and $\mathfrak{D}_i^{j,k,\ell}$ is given by

$$\mathfrak{D}_i^{j,k,\ell} = \begin{vmatrix} x_i & y_i & z_i & 1 \\ X_j & Y_j & Z_j & 1 \\ X_k & Y_k & Z_k & 1 \\ X_\ell & Y_\ell & Z_\ell & 1 \end{vmatrix} . \tag{F10}$$

If one wants to get an idea about the length of the summation (F9), one will find that there are

$$\rho_1\rho_2\rho_3 = C_3^{12}C_3^{9}C_3^{6} = \frac{12!}{3!9!}\ \frac{9!}{3!6!}\ \frac{6!}{3!3!} = 369{,}600 \text{ terms.}$$

The determinant $\Delta_{j_1,\ldots,j_{12}}$ is identically equal to zero as it was expected, and could be seen by performing the following rank-equivalent column-operations in Table E3, of Appendix E:

i) Add all the x-columns to one of them, say x_1-column

ii) Add all the y-columns to one of them, say y_1-column

iii) Add all the z-columns to one of them, say z_1-column

The resulting matrix has its first three columns zero. Thus

$$\Delta_{j_1,\ldots,j_{12}} = 0. \tag{F11}$$

Such an identity between invariants of the complete system is called a syzygy. Syzygies result from the expansions as above, of the determinants of the submatrices of dimension greater than six, until an admissible set of constraints is introduced which leave the jacobian matrix with six columns. By constraining the coordinates x_1,y_1,z_1,y_2,z_2,z_3, the jacobian for the case of four stations in range networks, (Table 1) is left with the columns of x_2,x_3,y_3,x_4,y_4,z_4.

Expanding the determinant of its 6x6 submatrix of the first six rows as above, one has

$$|\Delta|_{j_1,\ldots,j_6} = \sum_{\rho_1(\rho_2)} (-1)^{\sigma_1+\sigma_2} \begin{vmatrix} x_2 & 1 \\ x_{j_1} & 1 \end{vmatrix} \begin{vmatrix} x_3 & y_3 & 1 \\ x_{j_2} & y_{j_2} & 1 \\ x_{j_3} & y_{j_3} & 1 \end{vmatrix} \begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_{j_4} & y_{j_4} & z_{j_4} & 1 \\ x_{j_5} & y_{j_5} & z_{j_5} & 1 \\ x_{j_6} & y_{j_6} & z_{j_6} & 1 \end{vmatrix} \Delta_2^{j_1}\Delta_3^{j_2}\Delta_3^{j_3}\Delta_4^{j_4}\Delta_4^{j_5}\Delta_4^{j_6} \tag{F12}$$

Notice the presence of fundamental invariants of one dimension:

$$\begin{vmatrix} x_2 & 1 \\ X_{j_1} & 1 \end{vmatrix},$$

and two dimensions:

$$\begin{vmatrix} x_3 & y_3 & 1 \\ X_{j_2} & Y_{j_2} & 1 \\ X_{j_3} & Y_{j_3} & 1 \end{vmatrix}.$$

Expansion (F12) is of particular importance in the problem of determinantal loci for range networks.

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